On Propagation in Cylindrical Guides with Arbitrary Impedance Walls

HERBERT H. SNYDER
Department of Mathematics
Southern Illinois University, Carbondale, Illinois 62901

Abstract

In the context of guided wave propagation, the concept of *impedance wall* is an extension of the classical problem of propagation in guides bounded by perfectly conducting walls. The notion has been used by a number of workers for the study of some particular guide types in which the wall impedance may be assumed constant. The author formulates precisely the appropriate boundary value problem for general impedance walls for guides of constant circular section and, in particular, determines all axially symmetric solutions which yield waves propagating without attenuation. It was found that these solutions do not, in general, yield a set of complete orthogonal functions on the guide interior. When the analysis was appropriately modified to eliminate this difficulty, it was found that the theory is applicable to a much larger class of guides in which the wall impedance may be a non-constant function of distance along the guide.

Introduction

A prominent part of the classical theory of guided wave propagation is the theory of propagation in closed cylindrical tubes, of circular section, in which the wall is supposed perfectly conducting. When phrased as a steady-state boundary value problem for Maxwell's equations, the assumption of perfect conductivity reduces the condition: at the wall radius, the components of the electric vector $\overline{\mathrm{E}}$ tangent to the wall vanish identically. Analysis then leads to the well-known theory of modes of propagation in cylindrical guides. (See, for example, the developments of the theory in such treatises as (7, 11, 12 or 14).)

In recent years, some researchers have considered certain problems of guided wave propagation for cylindrical domains with the aid of a mathematical model in which the perfectly conducting cylindrical boundary is replaced by an *impedance wall.* For example, Unger (16, 17, 18) used such a model to study propagation in certain types of helical, lined and dielectric waveguides. As he stated in (16), such a model may be employed for waveguides which ". . . have a uniform and isotropic interior but an exterior which may be either heterogeneous, anisotropic or unlimited. To find the normal modes of wave propagation is considerably facilitated when the complex exterior region is replaced by an impedance wall at the interface to the interior. The boundary-value problem may then be formulated with the impedances of this wall."

I independently considered such problems in my search for reasonably simple mathematical models for the slow wave guides of traveling-wave tubes. To develop a useful *field theory* of phenomona in such devices, a model for the guide is required which may be applied to a variety of tube interiors: otherwise, an analysis must be carried out *ad hoc* for each one of the many interesting and potentially useful slow wave guide geometries.

In fact, a preliminary study along the lines just described was carried out by Birdsall and Whinnery (3). Although these authors clearly appreciate the desirability of having generally applicable field-theoretic models for such tubes, their analysis ". . . is primarily applicable to non-propagating structures such as the inductive- and resistive-wall amplifiers . . .". However, on the basis of the results given in this paper, I considered that the impedance wall concept is capable of yielding information of a much more general character. Most traveling-wave tubes consist of a cylindrical beam of electrons, surrounded co-axially by a slow wave guide, between which intervenes an empty cylindrical annular region. If wall impedances can be assigned—either at the outer boundary of the latter domain or at the boundary of the beam itself—so as to produce an interior field (in the absence of the beam) of the same character as that of an actual slow wave structure, then the possibly artificial character of the wall-impedance notion is a matter of indifference.

Accordingly, I adopt here the following point of view: the notion of wall impedance leads to a perfectly well-defined boundary value problem of electromagnetic theory, in its own right, of a more general character than the classical problem (which now appears as a simple special case). This problem was formulated precisely, and I studied the properties of the fields which emerged as a consequence.

Formulation of the Impedance-Wall Boundary Value Problem. Solutions in the Case of Constant Wall Impedance. Details for the Axially Symmetric Case.

Maxwell's equations for the interior of a vacuous region V are (in MKS units, (14))

$$\nabla \times \bar{H} - \epsilon \frac{\partial \bar{E}}{\partial t} = 0, \nabla \times \bar{E} + \mu \frac{\partial \bar{H}}{\partial t} = 0 \qquad [1]$$

where $\bar{E}$ and $\bar{H}$ are the usual electric and magnetic intensities, and ϵ and μ are the permittivity and permeability of vacuum, respectively. I was concerned only with single-frequency phenomena in the steady state, and thus assumed $\bar{E}$ and $\bar{H}$ to depend on the time in the form e^{-iwt} ($\omega = 2\pi f$, f an arbitrary positive frequency in cycles/second). Then, replacing $\bar{E}$, $\bar{H}$ in [1] by $\bar{E}\, e^{-iwt}$, $\bar{H}\, e^{-iwt}$ obtained the usual time-independent equations,

$$\Delta \times \bar{H} + i\omega\epsilon\bar{E} = 0, \Delta \times \bar{E} - i\omega\mu\bar{H} = 0 \qquad [2]$$

I took for region V, referred to cylindrical coordinates r, ø, z, the open cylindrical domain defined by $0 \leqslant r < B$, $0 \leqslant ø < 2\pi$, $/z/ < oo$, where the radius B of V is an arbitrary but fixed positive constant (Fig. 1).

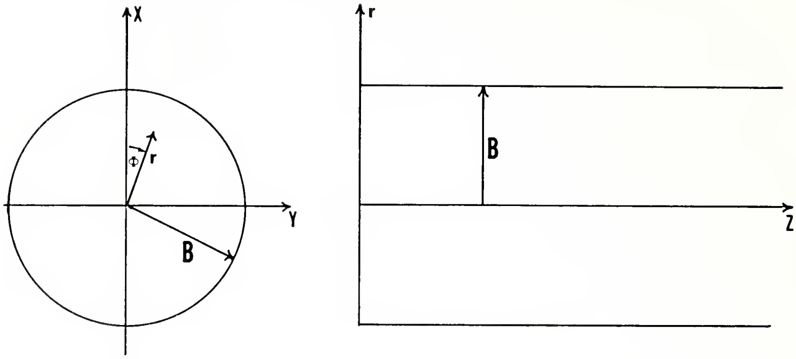

FIGURE 1. *From left to right, cross- and longitudinal sections of the domain V: $0 \leq r < B, |z| < \infty$. The wall impedance boundary conditions are applied at $r=B$.*

I therefore sought vectors $\overline{E}$ and $\overline{H}$ with the following properties: 1) at every point of V, $\overline{E}$ and $\overline{H}$ satisfy equations [2]; 2) these solutions have the character of waves traveling in the direction $z>0$; 3) at the boundary $r = B$, the angular and axial components of $\overline{E}$ and $\overline{H}$ satisfy the conditions

$$\frac{E_z}{H_\varphi} = -Z_1, \quad \frac{E_\varphi}{H_z} = Z_2 \qquad [3]$$

where Z_1 and Z_2 are complex-valued in general, but are independent of position (ϕ,z) on the wall $r = B$. (As seen below, [3] may be interpreted in such a way that Z_1,Z_2 represent *sequences* of complex numbers.

Following Unger in (16), Z_1, Z_2 were called the *impedances* of the wall $r = B$. The case where Z_1, Z_2 are independent of position nevertheless allows them to be arbitrary functions of the frequency parameter ω. This case was considered first, and then I showed how one may deal with problems for which the impedances may be functions of boundary position.

Since I intend to publish a fuller account elsewhere (13), the solution process was described only in outline and I proceed directly to the solutions themselves. The equations [2] were first written in the coordinates (r,ϕ,z). From the nature of the domain V, it was clear that the desired solutions must have period 2π in ϕ; supposing the field components to depend on ϕ in the form $\exp(in\phi)$ $(n = 0, \pm 1, \pm 2, \ldots)$, the equations were then reduced to equations in the n-th Fourier components of $\overline{E}$ and $\overline{H}$. Further, to satisfy condition 2), I supposed these latter components to vary with z as $\exp(i\beta z)$, for some (real) β to be determined. The resulting equations were then solved for the transverse components $E_r,E_\varphi,H_r,H_\varphi$ as linear functions of the axial components E_z,H_z and their derivatives with respect to r. Let $k = \omega/c$ ($c = 1/\sqrt{\mu\epsilon}$ being the usual plane wave phase propagation velocity). Two solutions were then obtained, according as $\beta > k$ or $\beta \leqslant k$. They were as follows:

$\beta > k$: slow waves

$$\left.\begin{aligned}
E_r &= \frac{\omega\mu n}{\Upsilon^2 r} B_n I_n(\Upsilon r) - \frac{i\beta}{\Upsilon} A_n I_n'(\Upsilon r) \\
E_\theta &= \frac{i\omega\mu}{\Upsilon} B_n I_n'(\Upsilon r) + \frac{\beta n}{\Upsilon^2 r} A_n I_n(\Upsilon r) \\
E_z &= A_n I_n(\Upsilon r) \\
H_r &= -\frac{i\beta}{\Upsilon} B_n I_n'(\Upsilon r) - \frac{\omega\epsilon n}{\Upsilon^2 r} A_n I_n(\Upsilon r) \\
H_\theta &= \frac{\beta n}{\Upsilon^2 r} B_n I_n(\Upsilon r) - \frac{i\omega\epsilon}{\Upsilon} A_n I_n'(\Upsilon r) \\
H_z &= B_n I_n(\Upsilon r)
\end{aligned}\right\} \times e^{i(n\phi + \beta z - \omega t)} \qquad [4]$$

$\beta \leqslant k$: fast waves

$$\left.\begin{aligned}
E_r &= -\frac{\omega\mu n}{\Upsilon^2 r} D_n J_n(\Upsilon r) + \frac{i\beta}{\Upsilon} C_n J_n'(\Upsilon r) \\
E_\theta &= -\frac{i\omega\mu}{\Upsilon} D_n J_n'(\Upsilon r) - \frac{\beta n}{\Upsilon^2 r} C_n J_n(\Upsilon r) \\
E_z &= C_n J_n(\Upsilon r) \\
H_r &= \frac{i\beta}{\Upsilon} D_n J_n'(\Upsilon r) + \frac{\omega\epsilon n}{\Upsilon^2 r} C_n J_n(\Upsilon r) \\
H_\theta &= -\frac{\beta n}{\Upsilon^2 r} D_n J_n(\Upsilon r) + \frac{i\omega\epsilon}{\Upsilon} C_n J_n'(\Upsilon r) \\
H_z &= D_n J_n(\Upsilon r)
\end{aligned}\right\} \times e^{i(n\phi + \beta z - \omega t)} \qquad [5]$$

where $n = 0, \pm 1, \pm 2, \ldots$, and

$$\begin{aligned}
\Upsilon &= \sqrt{\beta^2 - k^2}, \quad \beta > k \\
&= \sqrt{k^2 - \beta^2}, \quad \beta \leqslant k
\end{aligned} \qquad [6]$$

and $A_n, \ldots, D_n$ are constants of integration depending, in general, on k and n. The phase factor of the field, $\exp i(\beta z - \omega t)$, gave for the phase velocity $v = \omega/\beta$, and $v/c = k/\beta$ for the relative phase velocity; hence my designation as *slow (fast)* waves those with $\beta > k$ ($\beta \leqslant k$), which just corresponds to $v/c < 1$ ($v/c \geqslant 1$). In [5], $J_n(x)$ is Bessel's function of the first kind of order n (cf. (20)), and in [4], $I_n(x)$ is the modified Bessel's function of order n. In both [4] and [5], $()'$ denotes derivative with respect to the complete argument, Υr. I was interested in solutions with real values of β which—for either slow or fast waves—corresponds to propagation without attenuation in the direction $z > 0$.

It remained to impose the boundary conditions [3]. I recalled that [4] and [5] are expressions for the n-th term in the Fourier development of the field. In applying [3], I supposed that each such Fourier component has associated with it two impedances $Z^{(n)}_1$, $Z^{(n)}_2$. (Of course $Z^{(n)}_j = Z_j$ may be taken for all n, *i.e.*, it may be assumed in particular that each component "sees" the same impedance.) Calculation then gives, if $\beta > k$,

$$\left[I_n(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z^{(n)}_1 I'_n(\Upsilon B)\right] \left[\frac{i\omega\mu}{\Upsilon} I'_n(\Upsilon B) - Z^{(n)}_2 I_n(\Upsilon B)\right]$$

$$- \left(\frac{\beta n}{\Upsilon^2 B}\right)^2 Z^{(n)}_1 I^2_n(\Upsilon B) = 0 \qquad [7]$$

and if $\beta \leqslant k$,

$$\left[J_n(\Upsilon B + \frac{i\omega\epsilon}{\Upsilon} Z^{(n)}_1 J'_n(\Upsilon B)\right] \left[\frac{i\omega\mu}{\Upsilon} J'_n(\Upsilon B) + Z^{(n)}_2 J_n(\Upsilon B)\right]$$

$$+ \left(\frac{\beta n}{\Upsilon^2 B}\right)^2 Z^{(n)}_1 J^2_n(\Upsilon B) = 0 \qquad [8]$$

These are equations to determine β: Maxwell's equations [2] have in V the sequence of solutions [4] or [5] satisfying the boundary conditions 3), each unique up to a constant factor, provided that β satisfies [7] or [8], respectively. (I remark that—for the general case—if β and hence Υ^2 is actually a proper complex number, lying on neither axis of the complex plane, then it is largely a matter of indifference whether [4] or [5] is adopted as the canonical solution set.)

For solutions that are axially symmetric ($n = 0$), both solutions [4] and [5] and the equations [7] and [8] simplify radically. This results in

$$\beta > k$$

$$E_r = -\frac{i\beta}{\Upsilon} A_0 I_1(\Upsilon r) \qquad H_r = -\frac{i\beta}{\Upsilon} B_0 I_1(\Upsilon r)$$

$$H_Q = -\frac{i\omega\epsilon}{\Upsilon} A_0 I_1(\Upsilon r) \quad \text{(a)} \qquad E_Q = \frac{i\omega\mu}{\Upsilon} B_0 I_1(\Upsilon r) \quad \text{(b)} \qquad [9]$$

$$E_z = A_0 I_0(\Upsilon r) \qquad H_z = B_0 I_0(\Upsilon r)$$

and

$$\beta \leqslant k$$

$$E_r = -\frac{i\beta}{\Upsilon} C_0 J_1(\Upsilon r) \qquad H_r = -\frac{i\beta}{\Upsilon} D_0 J_1(\Upsilon r)$$

$$H_Q = -\frac{i\omega\epsilon}{\Upsilon} C_0 J_1(\Upsilon r) \quad \text{(a)} \qquad E_Q = \frac{i\omega\mu}{\Upsilon} D_0 J_1(\Upsilon r) \quad \text{(b)}$$

$$E_z = C_0 J_0(\Upsilon r) \qquad H_z = D_0 J_0(\Upsilon r) \qquad [10]$$

In place of [7] and [8] the following results for $\beta > k$

$$I_0(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z_1 I_1(\Upsilon B) = 0 \quad (A_0 \neq 0) \quad \text{(a)}$$

$$\frac{i\omega\mu}{\Upsilon} I_1(\Upsilon B) - Z_2 I_0(\Upsilon B) = 0 \quad (B_0 \neq 0) \quad \text{(b)} \qquad [11]$$

and for $\beta \leqslant k$,

$$J_0(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z_1 J_1(\Upsilon B) = 0 \quad (C_0 \neq 0) \quad \text{(a)}$$

$$-\frac{i\omega\mu}{\Upsilon} J_1(\Upsilon B) + Z_2 J_0(\Upsilon B) = 0 \quad (D_0 \neq 0) \quad \text{(b)} \qquad [12]$$

(where I wrote $Z^{(0)}_1 = Z_1$ $Z^{(0)}_2 = Z_2$). Thus if $n = 0$ (and only then), it was seen that the fields split into E- and H-waves, the coefficients A_0, B_0 (or C_0, D_0) are independent, and the value of Υ in [9a] satisfies [11a], while the Υ of [9b] satisfies [11b]: a similar remark holds for expressions [10] and equations [12].

The case of real values of $\beta > k$ was now considered. If I defined $Z_0 = \sqrt{\mu/\epsilon} \simeq 376.7$ ohms, then $\omega\epsilon = k/Z_0$ and $\omega\mu = kZ_0$ could be written. Then equations [11] could be written in the dimensionless forms

$$\frac{\Upsilon B I_0(\Upsilon B)}{I_1(\Upsilon B)} = kB \frac{iZ_1}{Z_0} \quad \text{(E-waves)} \quad \text{(a)} \qquad [13]$$

$$= kB \frac{iZ_0}{Z_2} \quad \text{(H-waves)} \quad \text{(b)}$$

respectively. The character of the solutions to [13] was readily seen by a simple graphical argument. The behavior of the common left-hand side of [13] is shown in Figure 2. It followed that equations [13] have unique real solutions ΥB (and hence βB) if, and only if, the right-hand sides were both real and $\geqslant 2$. In particular, it was obviously necessary that both Z_1, Z_2 be purely imaginary, with Im $Z_1 < 0$ and Im $Z_2 > 0$. In terms of wall impedances thus chosen, kB must be $\geqslant 2Z_0/iZ_1$ and kB must be $\geqslant 2Z_2/iZ_0$, respectively.

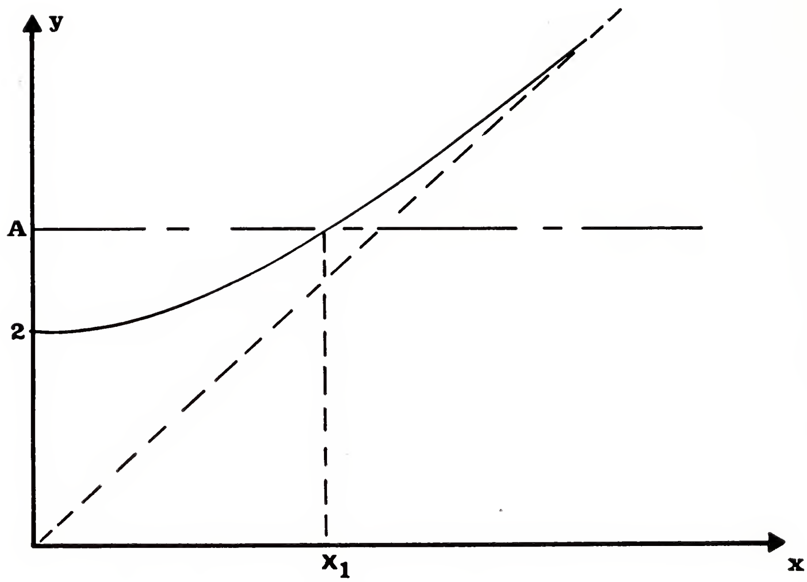

FIGURE 2. *Sketch of the graph of $y = xI_0(x)/I_1(x)$ for $x > 0$, from which it is clear that the equation $y = A$ has one and only one real solution x_1 (say) for every constant $A > 2$.*

However, *there was no sharp separation between the slow and fast wave situations*, as is described below. For, turning to equations [12], their dimensionless equivalents were obtained in the same way, *viz.*,

$$\frac{\Upsilon B J_0(\Upsilon B)}{J_1(\Upsilon B)} = kB\,\frac{iZ_1}{Z_0} \qquad \text{(E-waves)} \qquad \text{(a)} \quad [14]$$

$$= kB\,\frac{iZ_0}{Z_2} \qquad \text{(H-waves)} \qquad \text{(b)}$$

The behavior of $xJ_0(x)/J_1(x)$ for $x \geqslant 0$ is shown in Figure 3. The first vertical asymptote in Figure 3 occurs at the smallest positive zero of $J_1(x)$; letting x_1 be this zero, both equations [14] have infinitely many real solutions $\Upsilon B > x_1$ for all purely imaginary values of Z_1 and/or Z_2. There was, in fact, clearly a solution corresponding to each branch of the graph lying in the half-plane $x > x_1$. If $0 \leqslant \Upsilon B < x_1$, the equations [14] then had solutions if and only if the right-hand sides are $\leqslant 2$. Evidently it was only the last-mentioned situation which bore any relation to the real slow wave solutions already discussed.

For example, fixing upon a value of kB in $kB(iZ_1/Z_0)$, Z_1 was chosen so that $kB(iZ_1/Z_0)<2$. Then Figure 3 shows that [14a] has exactly one (fast wave) solution ΥB with $0<\Upsilon B<x_1$. If I made $kB(iZ_1/Z_0)$ increase, I obtained at $kB(iZ_1/Z_0) = 2$ the solution $\Upsilon B = 0$, satisfying both [13a] and [14a]. But now, for $kB(iZ_1/Z_0)>2$, [14a] clearly has no solution with $0<\Upsilon B<x_1$, while [13a] now has the single solution already discussed. As $kB(iZ_1/Z_0)$ increases in the manner described, then, the fast wave solution on $0<\Upsilon B<x_1$ transforms continuously into the slow wave solution of [13a]. However, no such transformation takes place with any of the infinitely many other fast wave solutions having $\Upsilon B>x_1$, bearing out my remark above regarding the inseparable nature of the fast and slow wave cases since, in general, it is seen that there will be no solution entirely free of fast waves. The existence and nature of the solutions to [13] and [14] has thus been revealed completely, in a qualitative way. (Explicit approximate formulas for these solutions were given in (13).)

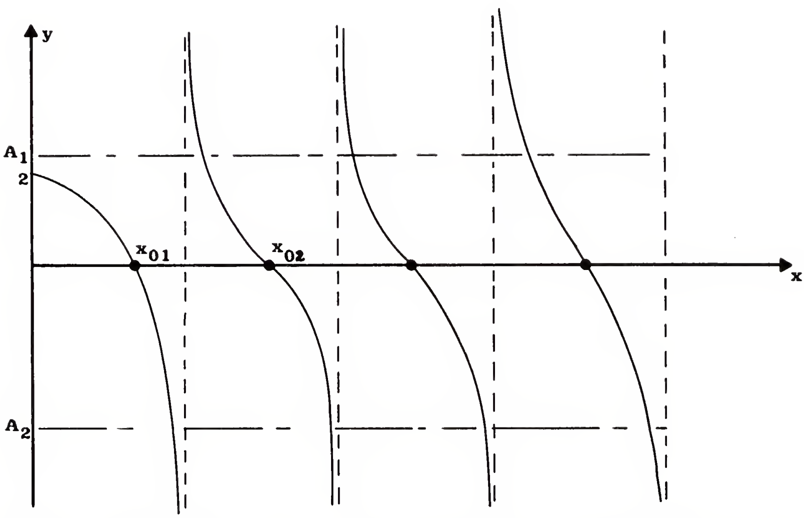

FIGURE 3. *Sketch of the graph of $y=xJ_0(x)/J_1(x)$ for $x>O$. The points x_0k are the zeros of $J_0(x)$. (The vertical asymptotes occur at the zeros x_1k of $J_1(x)$.) It is clear from the sketch that the equation $y=A$ always has countably many real solutions for every real A. If $A=A_2<2$, then there is a real solution on the interval $O<x<x_{11}$. If $A=A_1>2$, then there is no real solution on this interval.*

The total component E_z of the field with $kB(iZ_1/Z_0)>2$ is now considered. Then [13a] has the solution $\Upsilon B = X_0$ (say), and [14a] has the infinite sequence of solutions $\Upsilon_n B = X_n$ ($n = 1,2,3,\ldots$); Figure 4 shows these solutions. From [6], $\beta_0 B = \sqrt{(kB)^2 + X_0^2}$ (the slow wave) and $\beta_n B = \sqrt{(kB)^2 - X_n^2}$ (all the fast waves). From [9a] and [10a], E_z is now given by a sum over all these modes of propagation, *viz.*,

$$E_z = A_0 I_0\left(X_0 \frac{r}{B}\right) e^{i\beta_0 z} + \sum_{n=1}^{\infty} C_n J_0\left(X_n \frac{r}{B}\right) e^{i\beta_n z} \qquad [15]$$

Just as in the case of perfectly conducting wave guide, it is now seen that at a fixed but arbitrary value of kB, all but a finite number of the fast wave modes are beyond cut-off. However, a subtle aspect of the field not present in the classical case can now be seen: because of the presence of the slow wave component, $A_0 I_0(X_0 \frac{r}{B}) \exp(i\beta_0 z)$, the terms of the series [15] at $z = 0$, *viz.*,

$$A_0 I_0\left(X_0 \frac{r}{B}\right) + \sum_{n=1}^{\infty} C_n J_0\left(X_n \frac{r}{B}\right)$$

do not constitute a complete orthogonal set on V. It follows that the amplitude coefficients $A_0, C_1, C_2, \ldots$ cannot be determined by a Fourier-Bessel development of a prescribed initial field given at $z = 0$, as holds for the classical case. To determine these amplitudes, a technique must be employed which introduces the initial data at the outset. Such a method is furnished by the Laplace transform, which was applied next.

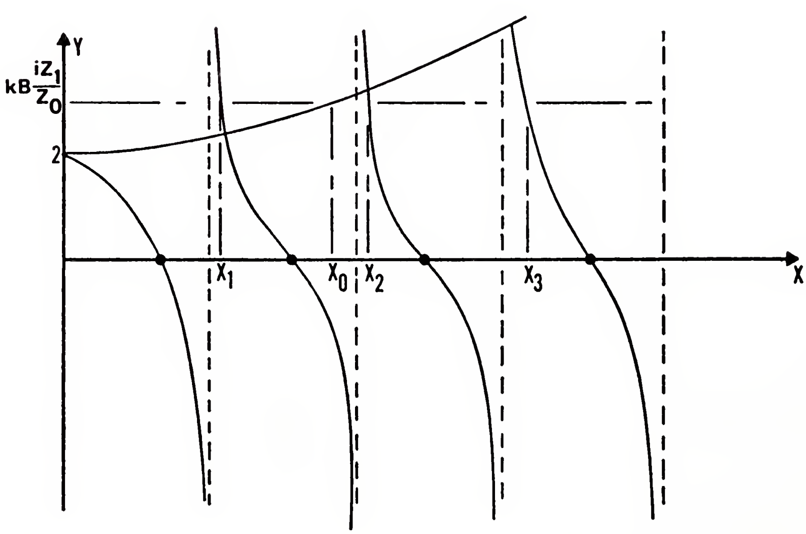

FIGURE 4. *This sketch indicates the location of the solutions of* $\Upsilon B I_0(\Upsilon B)/I_1(\Upsilon B) = kB(iZ_1/Z_0)$ *and* $\Upsilon B J_0(\Upsilon B)/J_1(\Upsilon B) = kB(iZ_1/Z_0)$ *for the case that* $kB(iZ_1/Z_0)$ *is real and* > 2. *The first equation has the single real solution* X_0, *the second an infinite sequence of solutions* $X_1, X_2, X_3, \ldots$, *of which the first three are indicated. In §§ 3 and 4, where* ΥB *takes complex values, the latter go over into the purely imaginary solutions* $iX_1, iX_2, iX_3, \ldots$.

Laplace Transforms of the Field Equations and Their Solutions

To overcome the lack of completeness observed in § 2, I returned to Maxwell's equations [2], reduced them again to equations in their n-th Fourier ϕ-components, and took their *Laplace transforms* with respect to z. If $F = F(r,n,z,\omega)$ is any one of the six reduced field components $E_r, \ldots, H_z$, the Laplace transform $\tilde{F}$ of F, with respect to z, is given by

$$L(F) = \tilde{F}(r,n,s,\omega) = \int_0^\infty e^{-sz}F(r,n,z,\omega)\,dz$$

if the integral on the right exists (see, *e.g.*, (5) or (9)). The transform enjoys *inter alia* the operational properties

$$L\left(\frac{\partial F}{\partial z}\right) = s\tilde{F} - F(r,n,0,\omega),\quad L\left(\frac{\partial^2 F}{\partial z^2}\right) = s^2\tilde{F} - sF(r,n,0,\omega) - \left.\frac{\partial F}{\partial z}\right|_{z=0}$$

When Laplace transforms of the reduced Maxwell's equations are taken, it is possible to solve for the *transforms* of the transverse components $\tilde{E}_r, \tilde{E}_\phi, \tilde{H}_r, \tilde{H}_\phi$ as linear functions of the *transforms* $\tilde{E}_z$, $\tilde{H}_z$ and their derivatives. In addition to $\tilde{E}_z$, $\tilde{H}_z$ and their derivatives, however, the values of the field components on the input plane $z = 0$ now enter these formulas explicitly. All details of the analysis were deferred to (13) and again I proceed directly to the solutions (writing down only those components which enter the boundary conditions [3] explicitly):

$$E_z = A_n(s)I_n(\Upsilon r) + P_n(r)$$

$$E_\phi = \frac{ins}{\Upsilon^2 r}A_n(s)I_n(\Upsilon r) - \frac{i\omega\mu}{\Upsilon}B_n(s)I'_n(\Upsilon r) - \frac{1}{\Upsilon^2}\left[\frac{ins}{r}P_n(r) - i\omega\mu\frac{d\Phi_1}{dr} + E_2(r)\right] \qquad [16]$$

$$H_z = B_n(s)I_n(\Upsilon r) + \Phi_n(r)$$

$$H_\phi = \frac{i\omega\epsilon}{\Upsilon}A_n(s)I'_n(\Upsilon r) - \frac{ins}{\Upsilon^2 r}B_n(s)I_n(\Upsilon r) - \frac{1}{\Upsilon^2}\left[i\omega\epsilon(dP_n/dr) + \frac{ins}{r}\phi^n(r) + H_2(r)\right]$$

In [16], $\Upsilon^2 = -(s^2 + k^2)$, Υ being that value of the root which reduces to $\beta^2 - k^2$ when $s = i\beta$, β being real with $|\beta| > k$. The quantities $P_n(r)$, $\phi_n(r)$, $E_2(r)$, $H_2(r)$ are explicit functions of the field components at $z = 0$, supposed known. The boundary conditions [3] were now applied to [16], which are written here in the form

$$\tilde{E}_z + Z_1^{(n)}\tilde{H}_\phi = 0,\quad \tilde{E}_\phi - Z_2^{(n)}\tilde{H}_z = 0 \qquad [17]$$

at r = B. When this was done, I obtained the coefficients $A_n(s), B_n(s)$:

$$A_n(s) = \frac{\left[\frac{i\omega\mu}{\Upsilon} I'_n(\Upsilon B) - Z_2^{(n)} I_n(\Upsilon B)\right] F_n(s) + \frac{ins}{\Upsilon^2 B} Z_1^{(n)} I_n(\Upsilon B) G_n(s)}{D_n(s)}$$

[18]

$$B_n(s) = \frac{\frac{ins}{\Upsilon^2 B} I_n(\Upsilon B) F_n(s) + \left[I_n(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z_1^{(n)} I'_n(\Upsilon B)\right] G_n(s)}{D_n(s)}$$

where, in [25],

$$D_n(s) = \left[I_n(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z_1^{(n)} I'_n(\Upsilon B)\right] \left[\frac{i\omega\mu}{\Upsilon} I'_n(\Upsilon B) - Z_2^{(n)} I_n(\Upsilon B)\right] + \left[\frac{ns}{\Upsilon^2 B} I_n(\Upsilon B)\right]^2 Z_1^{(n)}$$

[19]

$$F_n(s) = -P_n(B) + \frac{Z_1}{\Upsilon^2} \left[i\omega\epsilon \left.\frac{dP_n}{dr}\right|_{r=B} + \frac{ins}{B} \phi_n(B) + H_2(B) \right]$$

$$G_n(s) = Z_2 \phi_n(B) + \frac{1}{\Upsilon^2} \left[\frac{ins}{B} P_n(B) - i\omega\mu \left.\frac{d\Phi_n}{dr}\right|_{r=B} + E_2(B) \right]$$

It is precisely here—in the relations [17]-[19]—that the possibility of a far-reaching extension of the wall impedance notion is seen; since the other terms in [17] are the Laplace transforms of the field quantities, rather than the fields themselves, it followed that I was at perfect liberty to take for $Z_1^{(n)}$ and $Z_2^{(n)}$ not merely constant values, but *any functions of* s *for which the transforms possess inverses.* I thus allowed $Z_1^{(n)}$, $Z_2^{(n)}$ to be any analytic functions $Z_1^{(n)}(s)$, $Z_2^{(n)}(s)$, not necessarily constant, in all the general formulas of this section.

Expressions [16]-[19] now gave the Laplace transforms of all components of the field vectors. In particular,

$$\tilde{E}_z(r,n,s,\omega) = P_n(r) + \frac{\left[\frac{i\omega\mu}{\Upsilon} I'_n(\Upsilon B) - Z_2^{(n)} I_n(\Upsilon B)\right] F_n(s) + \frac{ins}{\Upsilon^2 B} Z_1^{(n)} I_n(\Upsilon B) G_n(s)}{D_n(s)} \quad [20]$$

and thus, integrating along the usual Bromwich path,

$$E_z(r,n,z,\omega) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \tilde{E}_z(r,n,s,\omega) e^{sz} ds \quad [21]$$

giving the n-th Fourier component of E_z, and h has the usual meaning for inversions of Laplace transforms. To obtain the total component E_z, [21] was multiplied by $\exp(in\phi)$ and sum over all integral values of n. From the formulas of this section, an altogether similar representation of H_z was obtained.

Some Remarks on the Inverse Transforms in the Axially Symmetric Case

When $n = 0$, the expressions simplify greatly. In particular,

$$E_z(r,z,\omega) = \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} \frac{F_0(s)I_0(\Upsilon r)e^{sz}ds}{I_0(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon}Z_1(s)I_1(\Upsilon B)} + \frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} P_0(r)e^{sz}ds \qquad [22]$$

We chose some particular values at $z = 0$ for which [22] assumes an especially simple form, the functions $F_0(s)$, $P_0(r)$ being then explicitly evaluable. Suppose that all components of the field vectors, together with their derivatives, vanish at $z = 0$ on $0 \leqslant r < B$, except for E_z only, and that $E_z(r,0,\omega) = E_0 = \text{const.}$ Then $E_1(r) = sE_0$, $H_2(B) = 0$, whereupon [22] becomes

$$E_z(r,z,\omega) = \frac{E_0B^2}{2\pi i}\int_{h-i\infty}^{h+i\infty} \frac{I_1(\Upsilon B)}{\Upsilon B}\,\frac{K_0(\Upsilon B) + \frac{i\omega\epsilon}{\Upsilon}Z_1K_1(\Upsilon B)}{I_0(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon}Z_1I_1(\Upsilon B)}\,I_0(\Upsilon r)e^{sz}s\,ds \qquad [23]$$

$$+ \frac{E_0}{2\pi i}\int_{h-i\infty}^{h+i\infty}\left[\frac{s}{\Upsilon^2} + \frac{\Upsilon BK_1(\Upsilon B)}{\Upsilon^2}\right] I_0(\Upsilon r)s\ \ e^{sz}ds$$

I concluded with a suitable *qualitative* discussion of [23], since the purpose of this special case (although of some technical interest in itself) was to bring out the main features of the propagation as directly as possible. Let us recall that Υ is now the function of s defined by $\Upsilon^2 = -(s^2 + k^2)$, made unique as already described above. Obviously, the branch points of Υ are $s = \pm ik$; are these also branch points of the integrands in [23]? First, $I_0(x)$ is an *even* entire function, and so $I_0(\Upsilon r)$ is an entire function of s for each r. The same is true of $I_1(x)/x$. Since $Z_1(s)$ is analytic, it follows that the only singularities of the first integrand of [23] due to $I_0(\Upsilon B) - \frac{i\omega\epsilon}{\Upsilon} Z_1(s)I_1(\Upsilon B)$ are poles. However, $K_0(x)$ and $K_1(x)$ both have logarithmic branch points at $x = 0$, hence the points $s = \pm ik$ are branch points of $K_0(\Upsilon B) + \frac{i\omega\epsilon}{\Upsilon}Z_1(s)K_1(\Upsilon B)$, and thus of the whole first integrand function.

For the second integral in [23], the points $\pm$ ik are both simple poles (of s/Υ^2) and branch points (of the second summand). A suitable contour for the integration is the path C_R shown in Figure 5, where I introduced the path $-\infty \pm ik$ as the branch cut. Thus C_R encloses neither branch point. For each of the integrals in [23], I wrote

$$\int_{h-iR}^{h+iR} = \int_{C_R} - \int_{C_{R'}} \qquad [24]$$

where $C_{R'}$ is that part of the contour other than the vertical segment joining the points h $\pm$ iR. As $R \rightarrow \infty$, the integral on the left in [24] approaches an integral in [23]. On the other hand, the first integral on the right in [24] gives the sum of the residues of the integrand at its poles. In the case $Z_1 =$ const. with kB $(iZ_1/Z_o) > 2$, these (simple) poles are the roots of [13a] already discussed. Taking the sum of residues at all these simple poles for the first integrand in [23], essentially the series [15] was obtained, *with the coefficients* $A_o, C_1, C_2,$ . . . *all determined uniquely.* However, more was obtained: the second integral in [23] yielded waves which propagate without attenuation at the velocity c, unaffected by the boundary conditions. There will also be contributions from both integrals due to the branch points, as the radius of the small circles drawn around each branch point was made to tend to zero. I conjectured that the role of the branch point contribution was essentially mathematical rather than physical, causing the solution to satisfy both Maxwell's equations and the boundary conditions on the *open* domain V': $z > 0$, *i.e.*, arbitrarily *near* $z = 0$. At some distance from the source plane $z = 0$, it was the residue series that was the physically significant part of the solution.

Conclusions. Some Further Remarks on the Literature

The impedance-wall boundary value problem has been formulated precisely for cylindrical domains. The solution of the axially symmetric, constant-impedance case was determined to within detailed examination of certain contour integrals in the complex plane. Those integrals represent inverse Laplace transforms, the introduction of which was necessary because of the lack of mode completeness on the entire open domain V. Contour integral representations for the general case were given. I found, in particular, that the Laplace transform allowed me to handle a class of problems in which the wall impedances are functions of position on the boundary $r = B$. A case of special technical interest is that in which the generalized wall impedances $Z_1^{(n)}(s)$, $Z_2^{(n)}(s)$ are Laplace transforms of *periodic* functions of z. On the basis of preliminary analyses already made by me, I conjecture that the solutions in such a case will have the properties of fields in actual periodic structures.

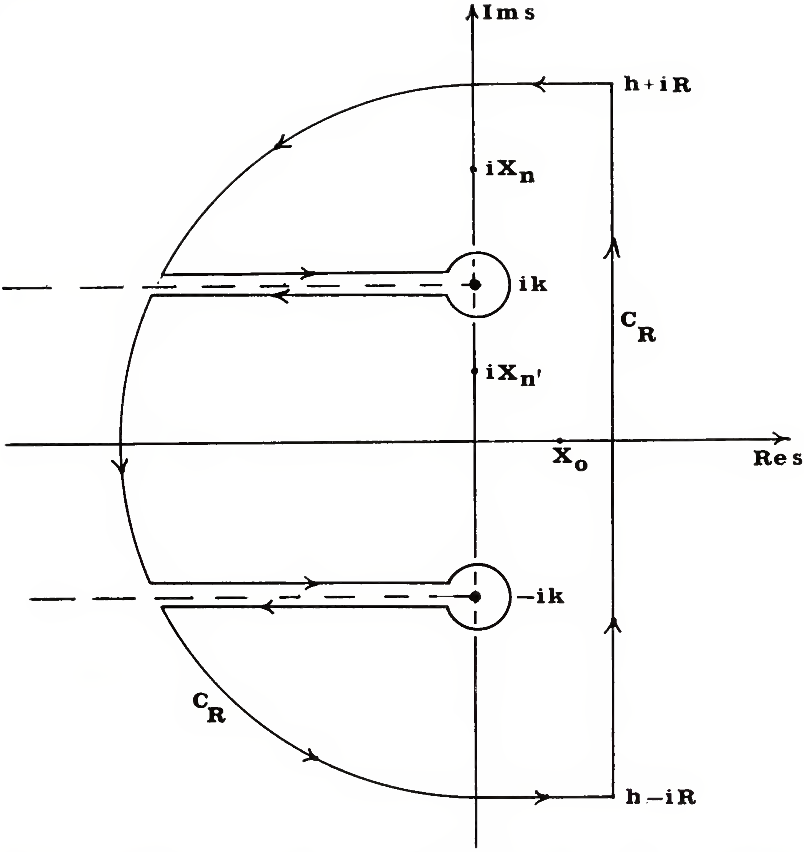

FIGURE 5. *The contour CR for inversion of the transform Ēz (r,s,w) in § 4. The contour avoids the purely imaginary branch points ±ik, as well as the simple poles iXn lying on the Im s axis. (iXn denotes those poles < k in absolute value.) In the limit as R→oo and the radii of the small circles about ±ik tend to zero, the right-hand side of [23] tends to the wanted field components Ēz (r,z,w).*

I wish to thank the referee for his constructively critical remarks, and to conclude with some observations relevant to the points raised by him.

I presume it is clear to the reader from the authors cited in my introduction that the notion of *surface impedance* is not original with me. Indeed, I do not know by whom the concept was first clearly formulated and used in the sense here employed, *viz.*, as a technique for approximate solution of boundary value problems in electromagnetic theory. I am, in fact, unaware of any published critique of the notion *per se*. The idea of an impedance manifold which, in some sense, guides or "supports" the propagation of electromagnetic waves, seems to be simply a tool used *ad hoc* for the approximate solution of problems, problems for which an exact solution may be unavailable or secured only at the expense of unjustifiably great labor. This *ad hoc* character becomes clear from the recent literature. Thus Bernardi and Valdoni (2) apply the notion of impedance wall ". . . to solve the problem of the propagation of TE zero-order modes in a rectangular waveguide, loaded by a thin and high dielectric constant slab against one side wall . . .". Tsandoulas (15) studied the character of a radiation field "due to a surface wave propagating along a rod waveguide having a surface impedance varying linearly with distance along the direction of propagation . . .". A similar application was given by Kritikos (8). The special nature of the problems considered is especially to be noted in all these cases. Bahar (1) and Gallawa (6) were chiefly interested in the development of impedance-wall models applicable to propagation between the earth and the ionosphere. Wait (19), in the same spirit, studied ". . . the modes which will be excited between two parallel [plane] impedance boundaries . . .". Savard (10) developed a theory ". . . in which the boundary conditions at [a cylindrical] guide surface are specified by an impedance dyadic . . ." for the study of certain surface-wave phenomena. It is clear that Savard has no such problems in mind as are here contemplated although some of my expressions bear a formal resemblance to his. Perhaps the best introduction to the notion of an impedance surface is to be found in the excellent expository article of Borgnis and Papas (4) who, without much ado, simply apply the notion to particular problems when and where it suited them.

My own exploitation of the notion of impedance wall is somewhat less special than that of our colleagues cited. Although I began by formulating the problem abstractly, in the interest of clarity of exposition, I consider that the chief merit of my work consists in pointing out that the impedance wall concept is capable of yielding models of propagation in guides for which the wall impedance is an "arbitrary" function of position on the (cylindrical) wall. As I have pointed out, my own interest lies mainly in the development of models for slow wave guides. However, it was my abstract formulation at the outset which led me to such a general conclusion.

Literature Cited

1. BAHAR, E. 1966. Propagation in a microwave model waveguide of variable surface impedance—theory and experiment. IEEE Trans. Microwave Theory and Tech. **MTT-15**:572-578.

2. BERNARDI, P., and F. VALDONI. 1968. The impedance wall concept applied to a dielectrically loaded waveguide problem. IEEE Trans. Microwave Theory and Tech. **MTT-16**:126-127.

3. BIRDSALL, C. K., and J. R. WHINNERY. 1953. Waves in an electron stream with general admittance walls. J. Appl. Phys. 24:314-323.

4. BORGNIS, F. E., and C. H. PAPAS. 1958. Electromagnetic waveguides and resonators. p. 285-401. *In* Handbuch der Physik XVI. Springer-Verlag, Berlin-Gottingen Heidelberg. 753 p.

5. CHURCHILL, R. V. 1944. Modern Operational Mathematics in Engineering, McGraw-Hill Book Co., N. Y. 206 p.

6. GALLAWA, R. L. 1964. Propagation in nonuniform waveguide with impedance walls. J. Res. Nat. Bur. Stds., Section D: Radio Sci. **68D**:1201-1213.

7. KRAUS, J. D. 1953. Electromagnetics. McGraw-Hill Book Co., New York, Toronto and London. 604 p.

8. KRITIKOS, H. N. 1965. Boundary waves along an impedance plane with a linearly varying impedance. IEEE Trans. Antennas and Propagation **AP-13**:577-583.

9. MCLACHLAN, N. W. 1939. Complex Variable and Operational Calculus with Technical Applications. Cambridge U. Press. 328 p.

10. SAVARD, J. Y. 1967. Higher-order cylindrical surface-wave modes. IEEE Trans. Microwave Theory and Tech. **MTT-15**:151-155.

11. SCHELKUNOFF, S. A. 1943. Electromagnetic Waves. Van Nostrand Co., New York, Toronto and London. 406 p.

12. SLATER, J. C. 1942. Microwave Transmission. McGraw-Hill Book Co., New York. 224 p.

13. SNYDER, H. H. 1971. Propagation in cylindrical domains bounded by arbitrary impedance walls. Electrical Commun. (In Press).

14. STRATTON, J. A. 1941. Electromagnetic Theory. McGraw-Hill Book Co., New York. 721 p.

15. TSANDOULAS, G. N. 1968. The effect of surface impedance variation on surface wave propagation along a rod waveguide. IEEE Trans. Microwave Theory and Tech. **MTT-16**:886-888.

16. UNGER, H.-G. 1962. Waveguides with anisotropic impedance walls. Electromagnetic Theory and Antennas. **6**:919-939.

17. ——————. 1958. Helix waveguide theory and application. Bell Syst. Tech. J. **37**:1599-1647.

18. ——————. 1962. Lined waveguide. Bell Syst. Tech. J. **41**:745-768.

19. WAIT, J. R. 1967. On the theory of shielded surface waves. IEEE Trans. Microwave Theory and Tech. **MTT-15**:410-414.

20. WATSON, G. N. 1958. A Treatise on the Theory of Bessel Functions. Cambridge U. Press. (2nd ed.) 804 p.