ASTRONOMICAL INTERPRETATIONS OF A FORMULA IN THE THEORY OF INTEGRAL INVARIANTS

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This article has been derived from an extensive manuscript by the writer, dealing with celestial mechanics. Although it is developed here as a contribution to the infinitesimal geometry of curves, another field of actualities to which it appertains is that of the permanences enjoyed by the orbit of a heavenly body while undergoing secular and periodic perturbations. Upon the latter subject Laplace* discourses as follows: "The permanency of the mean motions of the planets, and of the greater axes of their orbits, is one of the most remarkable phenomena in the system of the world. All other elements of the planetary ellipses are variable; these ellipses approach to and depart insensibly from the circular form; their inclination to a fixed plane or to the ecliptic augments and diminishes, and their perihelia and nodes are continually changing their places. These variations produced by the mutual actions of the planets on each other, are performed with such extreme slowness that for a number of centuries they are nearly proportional to the times."

A result proved by means of simple analysis in the paper referred to will now be stated. The ideal orbit C, of any revolving body solicited by a central force can always be obtained from the actual perturbed trajectory C′, by transforming the polar equation of the latter as follows:

$$T: r' = r + tP(r), \quad \theta' = \theta, \quad (t \doteq o),$$

in which the origin is the center of the attraction and

$$\sigma = tP(r) = ar^n + br^{n-1} + \cdots + l.$$

The polynomial tP(r) is determined by methods for curve fitting from n observations, (σ, r), of positions of the body upon C′, where the radius r′ of a point on C′ is connected with r, the intercept upon r′ by C, by the relation $r' = r + \sigma$. The coefficients a, . . ., l are small numbers, otherwise P(r) remains practically arbitrary, its coefficients changing as C′ ranges over the field of its admissible variations. The method holds for any central force.

Although the orbit C itself does not remain invariant under T, the mean motion and the greater axis (in the case of a planet) and an extensive system of other functions, some capable of simple geometric interpretation, remain unaltered. These functions are the quantities which stay permanent in the fluctuating orbital configuration.

Two of these invariant functions claim our attention. These constitute a subset of the whole system, unique for fundamental reasons. They are,

(1) $$A = \oint dr/P(r), \quad B = \theta_1 dr, \quad (\theta_1 = d\theta/dr).$$

One, viz. B, is universal for all C′; while A varies with C′. Referring to A, B as the characteristic invariants of T the problem which we propose in the following: *To determine all types of orbits C for which the characteristic invariants remain fixed numerically as C′ ranges over the field of perturbations.*

**Exposition du Systême du Monde*, (Transl. by Harte), V. 2, Book IV., Ch. 2, p. 23. (Laplace 1749-1827.)

Kirkwood, Proc. Amer. Phil. Soc., Vol. 22, (1885,) p. 424.

The necessary and sufficient condition is comprised in the invariant equation,

(2) $\phi dr/P(r) = \Theta_1 dr.$

Since r in P is the radius vector of points on C, the equation of segment C is

(3) $$\phi \int \frac{dr}{P(r)} = \Theta + \alpha.$$

The constant ϕ is arbitrary and may be chosen so as to simplify numerical coefficients in the relation (3). The elementary forms of this equation will now be enumerated. It is found that they include all stable orbital types that have been discovered in the universe, with telescope or by microscopic means.

(a) When P(r) turns out to be linear or linear to a high degree of approximation (3) may be reduced to the form,

$$r = e^{\theta+\alpha} + a.$$

The curve is, (a = o), a logarithmic spiral.*

(b) If P(r) is a perfect square we obtain

$$b(r+a)^{-1} = \theta + \alpha,$$

and the elementary form of the trajectory, (a = 0), is a reciprocal spiral.

(c) When P(r) takes the form of a perfect cube the orbit is

$$b(r+a)^{-2} = \theta + \alpha,$$

which gives the lituus spiral, (a = o).

As would be inferred from the forms of spiral nebulae neither the spiral of Archimedes nor the parabolic spiral, which have stop-points at or near the origin, can be obtained from (3).

(d) Let P(r) be a quadratic, (n = 2), whose factors are imaginary. The integral may be written,

(4) $$\mathrm{Cot}^{-1} \frac{r+a}{b} = \frac{1}{2}(\theta+\alpha),$$

and three very different cases are to be considered.

(I) First we assume that a, b, r, (a positive), are small infinitesimals; then a^2, b^2, r^2 are, in comparison, respectively, infinitesimals of the second order of magnitude. The formula (4) is equivalent to

(5) $$r = \frac{b^2/a}{1 - \cos\varphi} - \frac{r^2}{2a} - \delta,$$

where $\delta = (a^2+b^2)/2a$, $\varphi = \theta + \alpha$,
and this may be written,

(6) $$r + \delta = \frac{b^2/a}{1-(1-\epsilon)\cos\varphi} + \frac{1}{a}\left[\frac{b^2\epsilon\cos\varphi}{(1-\cos\varphi)^2} - \frac{r^2}{2}\right],$$

assuming ϵ to be sufficiently small and positive.

*Drawings of the spirals are given in many modern elementary books on the Differential Calculus, but comparatively little is known of the loci of polar equations containing parameters.

Now any ordinate τ of the cartesian arc,

$$y = b\,(\epsilon\cos\varphi)^{1/2} = \tau, \quad \left\{ |\varphi| > \beta > 0, \frac{\pi}{2} > \varphi > -\frac{\pi}{2} \right\},$$

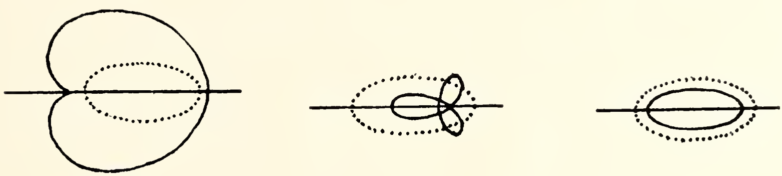
can be brought to differ by as small a positive quantity $h - \tau$ as may be desired from the corresponding ordinate of the line $y = h$ by taking h and ϵ small enough. Then with r sufficiently small the bracket of (6) will approach zero. In fact, from (5) the difference,

$$\frac{(b^2/a\sqrt{2})^2}{(1-\cos\varphi)^2} - \frac{r^2}{2},$$

is of the second order of magnitude, if we exclude from the values of φ a small region around zero. Hence since h becomes less than $b^2/a\sqrt{2}$ by choice of ϵ the bracket in (6) can be made negligible upon a segment C. Then (6) becomes,

$$(7) \qquad r + \delta = \frac{b^2/a}{1-(1-\epsilon)\cos\varphi}, \quad (|\varphi| > \beta > 0).$$

If δ is omitted this is a microscopic ellipse. With δ included three possible forms are shown in figure 1. This is the only derivation known to the writer of definite shapes appropriate for plane orbits of electrons.*



(II) Secondly, let us assume that r, in (4) remains very large. An equation (3) is then equivalent to

$$\frac{[(r+a)^2+b^2]^{1/2}}{r+a} = \sec\varphi.$$

We use λr instead of r, λ referring to the choice of linear units, Then,

$$(8) \qquad r = \frac{b - a\tan\varphi}{\lambda\tan\varphi}.$$

The hypothesis that r remains large is contradicted by its vanishing for

$$\varphi = \tan^{-1} b/a,$$

except in two cases. The first is when b is small and b/a approximately zero. Then the orbit is nearly circular, having an equation consecutive to $r = -a/\lambda$. The orbits of the planets are large ellipses approximating the circular form, that is with very small eccentricities.

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*J. H. Van Vleck, *Quantum Principles and Line Spectra*. Bulletin National Research Council No. 54; Ch. 5, 6, 7.

Next if a = o the equation (8) approximates to

$$(9) \qquad r = b/\tan \varphi, \left(\frac{\pi}{2} - |\varphi| > \beta > 0\right).$$

With m and e properly chosen this curve is consecutive to a conic

$$r = m/[1 - e \cos \varphi],$$

in the regions where r is large. In fact throughout its length (9), is a type of parabola which passes through the origin. The orbits of comets are parabolas or very long ellipses.

(III) Consider finally instead of (4), the equivalent

$$\tan 1 \frac{r+a}{b} = \Theta + \alpha,$$

in which r can be of any chosen magnitude. The elementary case is $r = \tan \Theta$. It represents a curve of two branches, passing though both the origin and the point infinity. This is not a stable orbit unless it be for an object falling into its center of attraction.