The Integrating Factor $R = x^m y^n$ for Certain First Order Differential Equations

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Consider a differential equation of the form

$$(h_1 x^{\alpha_1} y^{\beta_1} + k_1 x^{\gamma_1} y^{\delta_1})dx + (h_2 x^{\alpha_2} y^{\beta_1} + k_2 x^{\gamma_2} y^{\delta_2})dy = 0, \tag{1}$$

which is not exact. Suppose (1) may be made exact by means of the integrating factor $R = x^m y^n$. In order for $x^m y^n$ to be an integrating factor, certain linear relations must exist among the exponents of x and y, and, when these relations are satisfied, then certain other linear relations exist between m and n in which the h's and k's are coefficients. In general, the two equations in m and n are consistent, but under certain conditions these equations may be inconsistent. All these conditions are investigated in this study.

Assuming $x^m y^n$ to be an integrating factor, it follows that, if both members of (1) are multiplied by $x^m y^n$, the equation becomes exact, and then the partial derivative with respect to y of the coefficient of dx is equal to the partial derivative with respect to x of the coefficient of dy. Carrying through this multiplication, equating the partial derivatives, and factoring $x^{m-1}y^{n-1}$ out of all terms of the equation, one has

$$h_1(\beta_1+n)x^{\alpha_1+1}y^{\beta_1} + k_1(\delta_1+n)x^{\gamma_1+1}y^{\delta_1} = h_2(\alpha_2+m)x^{\alpha_2}y^{\beta_2+1} + k_2(\gamma_2+m)x^{\gamma_2}y^{\delta_2+1} \tag{2}$$

Since the members of (2) are identically equal, certain relations must exist among the exponents, and these appear to fall into two cases.

CASE I. $\alpha_1+1=\alpha_2, \beta_1=\beta_2+1, \quad \gamma_1+1=\gamma_2, \delta_1=\delta_2+1,$

or $\alpha_1+\beta_1=\alpha_2+\beta_2, \quad \gamma_1+\delta_1=\gamma_2+\delta_2$ (3)

Then $h_1(\beta_1+n)=h_2(\alpha_2+m), \quad k_1(\delta_1+n)=k_2(\gamma_2+m),$

or $h_2m-h_1n=h_2\alpha_2-h_1\beta_1, \quad k_2m-k_1n=k_1\delta_1-k_2\gamma_2$ (4)

The equations (4) may be solved for m and n provided $h_1k_2 \pm h_2k_1$. (5)

CASE II. $\alpha_1+1=\gamma_2, \beta_1=\delta_2+1, \quad \gamma_1+1=\alpha_2, \delta_1=\beta_2+1,$

or $\alpha_1+\beta_1=\gamma_2+\delta_2, \quad \alpha_2+\beta_2=\gamma_1+\delta_2$ (6)

Then $h_1(\beta_1+n)=k_2(\gamma_2+m), \quad k_1(\delta_1+n)=h_2(\alpha_2+m),$

or $k_2m-h_1n=h_1\beta_1-k_2\gamma_2, \quad h_2m-k_1n=k_1\delta_1-h_2\alpha_2$ (7)

The equations (7) may be solved for m and n provided $h_1h_2 \pm k_1k_2$. (8)

These two cases in reality are only one, for by interchanging the addends in the coefficient of dy and making the correction in signs, if necessary, Case I reduces to Case II, that is, abstractly there is only one case.

However, the first step to take in the solution of an equation of the form of (1) is to determine whether $h_1k_2=h_2k_1$ or $h_1h_2=k_1k_2$. If so, then no integrating factor of the form $x^m y^n$ may be determined. But then it is not necessary to determine such an integrating factor in order to find the solution of (1), for by making the substitution $y=vx$, the equation (1) reduces to a form whereby the variables may be separated, as will now be shown. The two cases will still be considered in order to avoid confusion in notation although the work will be carried out in only the one case since the method is the same.

CASE I. Assume $h_1k_2=h_2k_1$. Let $y=vx$. Then $dy=v\,dx+x\,dv$. Substituting in (1), imposing conditions (3), collecting like terms, and factoring, one finally obtains $\left[(h_1+h_2)v\,dx+h_2x\,dv\right]\left[h_1x^{\alpha_1+\beta_1}v^{\beta_1}+k_1x^{\gamma_1+\delta_1}v^{\delta_1}\right]=0$,

whence $(h_1+h_2)v\,dx+h_2x\,dv=0.$ (9)

Solving, one has $x^{h_1+h_2}v^{h_2}=C$, or finally $x^{h_1}y^{h_2}=C$ as the solution of (1).

CASE II. Assume $h_1h_2=k_1k_2$. Imposing conditions (6) and carrying out the work as before, one gets exactly the same result except that k_1 takes the place of h_1 in the coefficient of $v\,dx$ so that the diffierential equation becomes

$$(k_1+h_2)v\,dx+h_2x\,dv=0, \tag{10}$$

and the solution of (1) is $x^{k_1}y^{h_2}=C$.

In both cases the algebraic factor $h_1x^{\alpha_1+\beta_1}v^{\beta_1}+k_1x^{\gamma_1+\delta_1}v^{\delta_1}$ is obtained. When v is replaced by y/x, this factor becomes $h_1x^{\alpha_1}y^{\beta_1}+k_1x^{\gamma_1}y^{\delta_1}$, which is the coefficient of dx in (1). Hence, the coefficient of dy may be readily rearranged so as to contain the coefficient of dx in (1), and the resulting differential equation may then be written down by inspection. Thus, any differential equation of the form (1) in which the conditions (3) and (5) or (6) and (8) are satisfied may be solved by means of an integrating factor of the form x^my^n, or it may be solved by inspection when condition (3) or (6) holds but condition (5) or (8) does not hold, respectively.