PROGRAM OF THE SECTION ON MATHEMATICS

Chairman: C. K. Robbins, Purdue University

1. On the foundations of geometry. Karl Menger, University of Notre Dame.

J. E. Dotterer, Manchester College, was elected chairman of the section for 1938.

A Foundation of Projective Geometry

Karl Menger, University of Notre Dame

We assume a system of elements denoted by A, B, . . . and two operations which associate with any two elements A and B, an element $A+B$ and an element $A.B$. The operations may satisfy the following conditions:

I. Associativeness: $A+(B+C)=(A+B)+C$, $A.(B.C)=(A.B).C$.

II. Existence of two indifferent elements V ("vacuous" element) and U ("universal" element) such that for each element A we have:

$$A+V=A, \qquad A.U=A,$$
$$A.V=V, \qquad A+U=U.$$

III. A weakened distributive law:

$$A+(A+B).C=A+(A+C).B, \qquad A.(A.B+C)=A.(A.C+B)$$

From these assumptions we easily deduce that both operations are commutative and, for each A and B, satisfy the condition $A+(A.B)=A=A.(A+B)$. Consequently, if we have $A+B=A$ for two elements A and B, then we also have $A.B.=B$, and conversely. If for two elements A and$B$ both formulas $A+B=A$ and $A.B=B$ hold, then we call B a part of A. The part relation defined in this way has the ordinary properties. From C part of B, and B part of A, it follows that C part of A, etc.

We furthermore easily derive the formula $A+A=A=A.A$ for each A, by virtue of which there are neither multiples nor powers in the formulas of the algebra of geometry derived from the postulates I, II, III. We get thus: A part of A for each A; besides: V part of A, and A part of U on account of postulate II. Those elements which are different from V and do not contain any other parts than themselves and V shall be called points. Those elements which are different from U and are not part of any other element than themselves and U shall be called hyperplanes. Our definition makes precise the famous first words of Euclid's Elements: "Point is that which has no parts."

n points are called independent if for each of them the product of the point and the sum of the n-1 other points is V. In an analogous way, n hyperplanes are called independent if for each of them the sum of the hyperplane and the product of n-1 other hyperplanes is U. These definitions yield a theory of linear independence including in particular the following theorems: If an element is the sum of a finite number of points, then it may also be represented as the sum of independent points. If two systems of independent points have the same element A as sum, then the number of points in both systems is necessarily the same. We shall denote this number by a'. If an element is the product of a finite number of hyperplanes, then it may also be represented as the product of independent hyperplanes. If two systems of independent hyperplanes have the same element A as product, then the number of hyperplanes in both systems is necessarily the same. We shall denote this number a''.

If both A and B are elements which may be represented as the sum of a finite number of points, then the same holds for the elements $A+B=S$ and $A.B=P$, and we can prove that the number $a'+b'$ is not less than $s'+p'$. If both A and B may be represented as the product of a finite number of hyperplanes, then the same holds for S and P, and we can prove the number $a''+b''$ is not less than $s''+p''$.

In order to complete the theory, we need now two further postulates:

IV. If the point P is part of A, then there exists an element A' such that $P+A'=A$ and $P.A'=V$. If the element A is part of the hyperplane H, then there exists an element A'' such that $H.A''=A$ and $H+A''=U$.

V. A monotonic sequence cannot contain infinitely many different elements, i.e. if in the sequence $A_1, A_2, A_3, \ldots$, each element is part of the following element, or each element is part of the preceding element, then all elements of the sequence from a certain element on are identical.

These two postulates guarantee that each element A may actually be represented as the sum of a finite number of points, and as the product of a finite number of hyperplanes. Furthermore, Dr. F. Alt (Vienna) deduced from them that the number $a'+a''$ is the same for each element A, namely, equal to the number of independent points whose sum is U, and to the number of independent hyperplanes whose product is V. This formula together with the previously mentioned inequalities yields the formula $a'+b'=s'+p'$ for any two elements A and B.

We define now the dimension of an element A as the number of independent points whose sum is A, diminished by 1, thus dim $A=a'-1$, and set dim $V=-1$. For each element A the number dim A surpasses by exactly 1 the largest dimension of any part of A which is different from A. We have, furthermore, the formula:

dim A+dim B=dim $(A+B)$+dim $(A.B)$ for any two elements A and B.

Points are the elements of dimension O. If we call "straight line" each element of dimension 1, and say that a point P and a line L coincide if P is part of L, and if we add the postulate

VI. dim $U=n$,

then it is easy to deduce the ordinary postulates of projective geometry of the n-dimensional space, except the postulate that each straight line contains at least three distinct points. This postulate is easily seen to be equivalent to the following one:

VII. For any two distinct points P and Q there is an element A such that the distributive formula $A.(P+Q)=(A.P)+(A.Q)$ does not hold.

If we postulate, on the contrary, the distributive law

VII.* $A.(B+C)=(A.B)+(A.C)$ and $A.(B+C)=(A+B).(A+C)$ for any three elements A, B, C

(in which case the postulates I-V may be considerably simplified), then we get a Boolean algebra of classes. Under the assumption VI, the system of elements is isomorphic with the system of the 2^{n+1} subsets of a set containing exactly $n+1$ elements.