On a Class of Ruled Surfaces Generated by an Algebraic Correspondence

C. G. Schilling, Purdue University

Introduction. The (n,n)—Correspondence with complete symmetry between λ and μ was first set up by Emch[1] as follows:

$$(1)\quad \prod_{i=1}^{n+1}(\lambda-\lambda_i)\prod_{i=n+2}^{2n+2}(\mu-\lambda_i) - \prod_{i=1}^{n+1}(\mu-\lambda_i)\prod_{i=n+2}^{2n+2}(\lambda-\lambda_i=0.$$

In this relation the parameters λ and μ determine two geometric entities of the same kind uniquely. Furthermore, these parameters are related by (1) in such wise that an arbitrary choice for $\lambda(\mu)$ together with the n values thereby determined for $\mu(\lambda)$ form an involutorial set of $n+1$ numbers. Relation (1) represents a curve of order $2n$ and genus $(n-1)^2$.

In this investigation Zeuthen's formula[2]

$$(2)\quad y_1-y_2=2x_2(p_1-1)-2x_1(p_2-1)$$

is of importance. This formula applies when there exists between two carriers C_1 and C_2 of generi p_1, p_2 respectively, an $(x_1, x_2)-$ correspondence of such a nature that among the $x_1(x_2)$ points on $C_1(C_2)$ that correspond to one point on $C_2(C_1)$ it happens $y_1(y_2)$ times that two points coincide.

1. The Ruled Surfaces F_{4n}.

Let the parameter $\lambda(\mu)$ determine the planes of a pencil on generic line t and the parameter $\mu(\lambda)$ the osculating planes of a space cubic. If λ and μ are related as in (1), a surface F_{4n}, ruled and non-developable, of order $4n$ and genus $(n-1)^2$, is generated. From each point of the line t three planes osculating the cubic can be drawn, and, due to the continuity of (1), all these planes will be determined. To each plane corresponds n lines which lie in it. Line t is therefore a $3n$-fold line on the surface. Hence the $3n$ lines through any point of line t lie in three planes, n lines in a plane.

Between the points of line t and of the space cubic there is a (1,3)-correspondence. Then by Zeuthen's formula, (2) above, there are four coincidences. These are due to the four points where the line t intersects the developable on the cubic. The line t intersects four lines on this developable, and these lines are formed by the intersection of two consecutive osculating planes of the cubic. Hence, there are n torsals at each of these four points, a torsal point being one in which two consecutive generatrices of the surface intersect.

The work of Wiman[3] is useful in any investigation of the fundamental characteristics—multiple curve, torsals—of a ruled surface. The formulas we apply in the following are due to Wiman. The number of torsal points on the $3n$-fold directrix is

$$2(r+p-1)=2n^2+2n$$

[1]Emch, 1932. Amer. Jour. 54(2):285.
[2]Zeuthen, 1871. Math. Ann. 3:150.
[3]Wiman, Acta Math., 1895-7. 19-20:63.

where p is the genus of the surface and r the multiplicity of the directrix. The number of torsal points on the surface outside of those on the directrix line is

$$2(n-r+p-1)=2n^2-2n$$

making a total of $4n^2$ torsal points on the surface.

We wish to investigate what effect the roots of $\triangle=0$, where $\triangle$ is the discriminant of (1), have on the surface F_{4n}. The equation $\triangle=0$ is of degree $2n(n-1)$ in $\lambda(\mu)$, and its roots separate into $2n$ sets of $(n-1)$ numbers each, each set giving the same double root. The two sets of values, the λ-set and the μ-set, given by (1) are indistinguishable. Hence the roots of $\triangle=0$ represent both sets. First, consider these roots as μ-values, and let line t carry the μ-planes and the cubic the λ-planes. Then each of one of the above named sets of $(n-1)$ μ-planes on t determines, among others, the same two consecutive λ-planes on the cubic. Thus are formed torsal points which lie on a line of the developable on the cubic. Clearly there are $2n^2-2n$ such torsal points since there are $2n$ such sets of $(n-1)$ roots of $\triangle=0$. If the roots of $\triangle=0$ are considered as λ-values then $2n^2-2n$ torsal points are formed on the $3n$-fold directrix line t.

These same results might have been obtained by using Zeuthen's formula (2) on the $(1,n)$-correspondence that exists between the points on either carrier and the lines on F_{4n}. We observe that $4n$ of the torsal points on the surface are not accounted for by the roots of $\triangle=0$. These are independent of the relation (1) and are determined by the choice of line t. They always occur in sets of n at the four points where line t strikes the developable on the cubic. They are part of the multiple curve, the $3n$-fold directrix, but the residual double curve does not pass through them.

The torsal points of the surface not on directrix line t lie on $2n$ lines, $(n-1)$ on a line. The torsal points of the surface on line t are so arranged that the torsal lines on t lie in sets of $(n-1)$ each in $2n$ planes on t.

Through the line t pass $3n$ sheets of the surface in such a manner that the $3n$ lines through any point of t lie n at a time in three planes. Exceptions occur only for the roots of $\triangle=0$, causing two of the planes to coincide thereby producing torsals as already found. Hence the surface has no double lines or stationary lines.

A generic section of the surface is a curve of order $4n$ and genus $(n-1)^2$. It has $7n^2-4n$ nodes, $\frac{1}{2}3n(3n-1)$ of which are due to the $3n$-fold directrix line on the surface. Hence, there is a residual double curve on the surface of order $\frac{1}{2}5n(n-1)$. By a formula due to Wiman the genus of the residual double curve is

$$\begin{aligned} p' &= \tfrac{1}{2}(n'-r-2)\ (n'-r-3)\ +\ p(n'-r-2)-D^* \\ &= \tfrac{1}{2}(n-2)\ (2n^2-3n-1)-D. \end{aligned}$$

The number D (actual double points on the double curve) is increased by 2 for each double line on the surface and by 1 each time that two torsal lines cross. Since neither possibility occurs for the surfaces F_{4n}, $D=0$ and the genus of the residual double curve is

$$p'=\tfrac{1}{2}(2n^3-7n^2+5n+2).$$

II. The Rational Case.

In this case line t is a triple line and on it are four torsal points where this line intersects the developable of order 4 on the twisted cubic. There is no residual double curve.

*In this formula n' denotes the order of the ruled surface.

We now proceed to map a plane into a rational, ruled quartic surface in S_5 and then to project this surface into S_3 thereby obtaining the above determined ruled quartic. This method of examining a surface, by first obtaining it as a normal surface, is useful in viewing the special properties of a surface, such as the double curve, torsals and multiple points, if any.

In the plane π take a system of rational quartics with a fixed triple point and three fixed single points. There are ∞^5 such quartics, and any two members of the system intersect in four free intersections. Now map the plane π by means of this system into an S_5 as follows:

(3) $\sigma y_i = \phi_i(x), \qquad i=1,2,3,4,5,6$

where the $\phi_i(x)$ are the six linearly independent quartics on the four base points. Then in S_5 is obtained a rational, ruled, normal quartic surface F. There are two types of surface F in S_5, (a) one with an ∞^1 system of directrix conics and the other (b) with a directrix line. Mapping by (3) gives type (a). A base point of order k in plane π maps into a rational curve of order k in S_5. Then the base points of the system (3) map into corresponding rational curves To the sections of F by hyperplanes correspond projectively the curves of the system (3). Then all sections of F by hyperplanes are rational and the surface F must necessarily be ruled.

The surface F has ∞^1 directrix conics; any such conic is determined by one point of F and no two conics intersect.[5] To understand this statement consider the ∞^1 system of conics in π on the four base points. A conic in π maps into a curve of order 8 but from this octic splits off three lines and a cubic leaving a conic on F. There are ∞^1 of these on F, as in π, and each is determined by one point on F, just as in π one point in the plane determines a conic of this pencil, and this one point maps into just one point on F. No two conics on F intersect since no two conics of the pencil intersect in any points outside of the base points.

This surface F also contains ∞^3 directrix cubics any such curve being determined by three points of F. All these can be obtained as hyperplane sections of F residual to any given generator. This statement is clear when we consider in plane π the rational cubics through the four base points and with a double point at the triple base point. By the mapping a cubic goes into a C_{12} but from this twelve-ic splits off two cubics and three lines thus leaving a cubic. Since any three points in π outside of the base points determine one of these cubics in π, then any such directrix cubic is determined by three points on F. In the plane π any two of these cubics intersect in two free points, hence on F any two of the directrix cubics do likewise. There are ∞^3 such cubics in π, hence that many on F. This normal surface in S_5 can now be projected from a line in S_5, and the ruled quartic we have studied in S_3 is obtained.

III. The Dual Surfaces. Γ_{4n}.

Let the parameters λ and μ in (1) determine points on the generic line t and the twisted cubic C_3 in S_3. Then a ruled surface of order $4n$ and genus $(n-1)^2$ is generated by joining corresponding points. Line t and the twisted cubic are n-fold directrices on the surface. A generic cross section of this surface is a curve of order $4n$ and genus $(n-1)^2$. It, therefore, has $7n^2-4n$ nodes, of which $2n^2-2n$ are due to the four n-fold points caused by the two directrices. Hence, the surface has a residual double curve, call it the b-curve, of order $5n^2-2n$.

[5]Edge, 1931. Ruled surfaces. Cambridge University Press.

Each generator of the surface is cut by $(4n\text{-}2)$ others of which $(n\text{-}1)$ are at each of its points on a directrix, thus discarding $2n\text{-}2$ in all. Then the residual b-curve of order $5n^2-2n$ is generated by the $(4n\text{-}2)-(2n\text{-}2)=2n$ points, in which each generatrix is cut outside the directrices. As before, the roots of $\triangle=0$ may be considered as λ- or μ-values and in each case $(2n^2\text{-}2n)$ torsal points are determined on one of the directrices. The surface has $4n$ additional torsals, as given by the general formula. These arise, n at a time, at the four points where the line t intersects the developable of order 4 on the cubic. They are on the twisted cubic and the residual b-curve does not pass through them.

As in the case of its dual F_{4n}, this surface has no double lines, and therefore no stationary lines. A section of Γ_{4n} by any plane through t is a curve of order Γ_{4n} consisting of an n-fold line and three sets of n concurrent lines. Hence, there is a $(1,3n)$-correspondence set up between the planes on t and the lines on $4n$. By Zeuthen's formula (2) there are $(2n^2+2n)$ coincidences of lines. These coincidences are torsal points on the directrix C_3, and we have already observed that along this cubic there are $2n^2-2n+4n=2n^2+2n$ torsal points. Any plane on t cuts the residual b-curve in $(5n^2-2n)$ points. In this plane are 3 sets of concurrent lines, n lines in a set. Outside of the intersections at the vertices of the pencils, these lines intersect in $\frac{1}{2}3n(3n\text{-}1)-3\cdot\frac{1}{2}n(n\text{-}1)=3n^2$. That is to say, any plane on line t cuts the b-curve in $3n^2$ points outside of line t hence $2n^2\text{-}2n$ points of the curve are on line t. These are the torsal points on t as already found.

The genus of the residual b-curve of order $(5n^2\text{-}2n)$ can be determined by the correspondence between the points of the b-curve and the lines of the surface. Each point of the curve corresponds to the two generatrices through it, and each generatrix passes through $(4n\text{-}2)-(2n\text{-}2)=2n$ points of the b-curve, hence, a $(2n,2)$-correspodnence between points on the curve and lines on the surface. No torsal points of the surface occur off the two n-fold directrices, hence no coincidences in this correspondence. Now apply Zeuthen's formula (2):

b-curve: $x_1=2n,\ y_1=0,\ p_1=?$

Surface: $x_2=2,\ y_2=0,\ p_2=(n\text{-}1)^2.$

Then $p_1=n^3-2n^2+1=(n\text{-}1)\ (n^2-n-1)$, and this is the genus of the residual double curve. As before, the rational surface of this class can be obtained by first mapping the plane into a quartic in S_5 with a directrix line.