THE VALIDITY AND INTERPRETATION OF MATHEMATICAL FORMULAS.

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At the last annual meeting at Philadelphia of the American Association for the Advancement of Science an interesting address was given by Dr. J. E. Adams of the University of Kentucky on the subject "Is Education Antiquated?" Dr. Adams carried out his investigations by making a survey of high school pupils in Kentucky and received answers from 5,000 pupils to such questions as: "What were the subjects in which you failed? What subjects would you like taken out of the curriculum, and what subjects would you like to take that you cannot get?" The following is a summary of the results as stated by *Science Service:*

"It was found that mathematics and Latin were responsible for far the greater percentage of the failures. The difficulty and lack of practical application of these subjects made them very unpopular. Algebra, geometry and Latin were wanted out of the curriculum. Of the subjects which the pupils wanted to take and could not get, the girls agreed upon domestic science, French, Spanish, typewriting and bookkeeping, and the boys upon manual training, chemistry, typewriting and bookkeeping."

In a recent issue of *The Terre Haute Star*[1] was printed an editorial entitled "Maximums and Minimums in 'Math.'", in which the writer makes an attack upon the rules requiring students to take any mathematics beyond arithmetic. Despite his also apparent antipathy for Latin, the following extract from his criticism is quoted: "Pupils who show an aptitude and a love for other studies, to them certain to be useful studies, are compelled to put in three or four hours daily on home work on a study which they don't understand, never can understand and which they loathe."

These are two illustrations of a change in sentiment that is taking place concerning mathematics, and doubtless louder complaints will be heard in the future, for the heavy scholarship mortality rate among engineering students in American universities can be traced in part to the failure of so many students to pass their mathematics courses. This is shown by a report[2] of a committee of The Society for the Promotion of Engineering Education which states that judging from the past results only about 30 per cent of an entering Freshman class can be expected to complete the engineering courses in the prescribed four years.

The question then is: What are the mathematics teachers going to do about it? Certainly the question is not easily answered and this dis-

[1] Terre Haute Star, August 31, 1927.

[2] Journal of Engineering Education, September, 1926, pp. 121-127.

"Proc. Ind. Acad. Sci., vol. 37, 1927 (1928)."

cussion will be confined to offering a few suggestions to college instructors which it is hoped will help them to make mathematics more interesting and intelligible to college students.

One of the tendencies in mathematics teaching is to reduce laws and processes to formulas. The student learns rules and mechanical ways for solving certain type problems and generally deludes himself into believing that successful manipulation of these formulas constitutes a knowledge of mathematics. Certainly this constitutes the path of least resistance for both student and instructor. However, it is obvious that a mere manipulation of formulas does not imply any real, fundamental knowledge of the subject any more than the manipulation of the controls of an automobile implies that one knows anything about their construction and theory. In other words, the student works with formulas about whose validity and interpretation he knows little or nothing. And this criticism need not be confined to class room students for it applies equally well to many hand-book users. One reason for this state of affairs can be charged to the textbooks which are for the most part written from the standpoint of abstract mathematics, from the classical standpoint.

The formula in mathematics represents the condensation of a more or less generalized process into a few symbols, but these symbols may be open to many interpretations. It has been well said that one gets no more out of a thing than one puts into it, but frequently a formula may have numerous interpretations. The sentence: "St. Clair St. is a st. st." with *Saint*, *street* and *straight* all represented by the abbreviation *st.*, illustrates how the same symbol has a number of different interpretations. Likewise a mathematical formula may have numerous interpretations as for example the integral $\int x^n dx$ which has an infinite number of interpretations although so far only a few have any practical value. As a special case the integral $\int_a^b x^3 dx$ may represent the area under the curve $y=x^3$ between the limits a and b, or it may represent the first moment with respect to the y-axis of a part of the area under the parabola $y=x^2$, or it may represent the second moment or moment of inertia of the area under a part of the straight line $y=x$. And in curve fitting for statistical purposes this same integral might represent third, fourth and higher moments.

Again the integral $\int x^3 dx$ could be interpreted as a line integral with the x-axis as the path of integration, or it might be considered as a line integral $\int x^n y dx$ where the integration is along the path $y=x^{3-n}$. Certainly this simple integral shows the importance of interpretation, for the mere abstract integration is very easy.

The integration of the above gives $x^4/4$ between the limits a and b with the final result $(a^4-b^4)/4$. What does this mean? Few students ever see that if one writes $y=x^4/4$, where y stands for the area, first moment, moment of inertia, or whatever interpretation is put on the integral, the value of the integral is found as the difference of two ordinates. That is, one evaluates the integral as the difference of two ordinates but interprets the results as units of area or something else, just as one reads mere scaled units on a thermometer but interprets the reading as temperature.

The preceding results clearly show the necessity of using extreme care in interpreting a formula. Mathematical physics has numerous formulas of broad application whose interpretation depends upon the particular branch of physics in which the formula is applied. Thus, if the divergence equals zero at a given point, it implies in heat no increase or loss of heat at that point, in electricity no change in flux at the point, and in liquids incompressibility. Likewise if the curl of a system of forces is zero over a field, then the system of forces is a conservative system over that field, or if applied to motion, the motion is irrotational. As an abstract relation, if the curl of a function is zero for a given region the line integral is independent of the path in the region, and if the curl for a differential equation of the form $Mdx+Ndy=0$ is zero, the differential equation is exact.

What has all this to do with making mathematics more intelligible to students? It means that teachers of mathematics must attempt where ever possible to interpret formulas in terms of every-day uses or physical results. More emphasis must be put on the interpretation of the formula and less on the mechanics of solutions.

As to the validity of formulas, it is easy to find formulas that are valid only under certain conditions. Thus, for example, the formula for fluid pressure is frequently given as $P=w\int xy\,dx$, but this formula is valid provided the origin is chosen at the surface of the fluid, which is not always the most advantageous. Accordingly, the best way to present the method of calculating fluid pressure is to build the formula for the problem at hand using only the physics of the problem as a guide. Many other such illustrations could be given, but only one more will be cited, namely the well known statistical formula for the computation of the standard deviation of a sum. Frequently this formula is used without giving any thought to the fact that its validity depends on a large number of measurements, and it is applied to a small number of items with the results deduced consequently being questionable. This same statement can be made concerning the use of other formulas in statistics. Some of the formulas of statistics are derived from the properties of the symmetrical probability curve and hence are really only valid when the data to which they are applied are symmetrical or only moderately skewed. This fact is frequently overlooked. The application of a formula implies the validity of its use, and the teacher and student must observe and emphasize this phase of its use. In other words the formula must not become a mere mathematical abstraction but a vital, working process whose derivation is clearly understood and the assumptions underlying it must also be clearly understood. Textbooks must be written with this aim in view and teachers of mathematics in the future must have a broader preparation in physics and chemistry.

The beauty of pure mathematics is not lost, but rather enhanced, by the fact that, while it does not depend on the physical world for an interpretation, it may be possible to interpret it in terms of the practical world in which one lives. The very fact that nearly all of high school mathematics and much of college mathematics is reduced, so it seems to the student, to the manipulation of oftentimes meaningless formulas which are merely remembered and used blindly by the student, without any connections with his past experiences, makes the subject seem mysterious, fearful, uninspiring and difficult. Of course references are frequently made to applications and occasionally a simple application is discussed in textbooks, but the application side of formulas is generally abstract, analytical or geometrical, and superficial so far as the experiences of the student are concerned. The application side of mathematics should be developed from the very beginning and the student must be made to feel and realize that he is taking mathematics for more than the cultural development it may give him. This attitude implies no compromise on the value of pure abstract mathematics, for its value is just as great as most of the other subjects in the curriculum. Neither should the mathematics courses be reduced to mere application courses. The great German mathematician Felix Klein has well expressed the ideal which should govern mathematical teaching in the following statement:

"First as concerns the *success* of teaching mathematics. No instruction in the high school is as difficult as that of mathematics, since the large majority of students are at first decidedly disinclined to be harnessed in the rigid framework of logical conclusions. The interest of young people is won much more easily, if sense-objects are made the starting point and the transition to abstract formulation is brought about gradually. For this reason it is psychologically quite correct to follow this course.

"No less to be recommended is this course if we inquire into the essential purpose of mathematical instruction. Formerly it was too exclusively held that this purpose is to sharpen the understanding. Surely another important end is to implant in the student the conviction that *correct thinking based on true premises secures mastery over the world.* To accomplish this the outer world must receive its share of attention from the very beginning.

"Doubtless this is true but there is a danger which needs pointing out. It is as in the case of language teaching where the modern tendency is to secure in addition to grammar also an understanding of the authors. The danger lies in grammar being completely set aside leaving the subject without its indispensable solid basis. Just so in the teaching of mathematics it is possible to accumulate interesting applications to such an extent as to stunt the essential logical development. This should in no wise be permitted, for thus the kernel of the whole matter is lost. Therefore: We do want throughout a quickening of mathematical instruction by the introduction of applications, but we do not want that the pendulum, which in former decades may have inclined too much toward the abstract side, should now swing to the other extreme; we would rather pursue the proper middle course."[3]

[3]Memorabilia Mathematica, Moritz, p. 77.