Some Relations of Plane and Spheric Geometry.

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Our notions of *plane analytic geometry* date to the publication by Descartes of his philosophical work: "*Discours de la méthode . . . dans les sciences*," 1637, which contained an appendix on "*La Geometrie*." In this work Descartes devised a method of expressing a plane locus by means of a relation between the distances of any point of the locus from two fixed lines. This discovery of Descartes led to the analytic geometry of the plane, and the extension to three dimensional space gave rise to geometry of space figures by the analytic method. A single equation, $f(x,y) = 0$, between two variables represents a plane curve; a single equation, $F_1(x,y,z) = 0$, in three variables represents a surface in space; and two equations, $F_1(x,y,z) = 0$, $F_2(x,y,z) = 0$, represent a curve in space.

In the Cartesian system of coördinates, a space curve is determined by the intersection of two surfaces. If we wish to investigate the curves upon a single surface, that is, if we wish to devise a geometry of a given surface, it may be possible to discover a system of coördinates upon the surface, such that any surface-locus may be expressed by a single equation in terms of two coördinates, as in plane geometry. The sphere furnishes a simple example in which a locus upon its surface may be represented by a single equation connecting the coördinates of any point upon the locus.

Toward the end of the eighteenth century a fragmentary system of analytic geometry of loci upon the surface of the sphere was developed. This early work on *Spheric Geometry* seems to have originated with Euler (1707-1783), but many of the special cases of spherical loci were investigated by Euler's colleagues and assistants at St. Petersburg. In the present paper are enumerated a number of the early investigations on spherical loci, and a derivation of the equations of sphero-conics in modern notation. The correspondence of the *spheric equations* to the similar equations of plane analytics is shown.

Historical.

One of the first problems involving a locus upon a sphere to be solved by use of spherical coördinates was the following: *Find the locus of the vertex of a spherical triangle having a constant area and a fixed base.* With the base AB fixed, Fig. 1, and the area of the spherical triangle APB constant, the

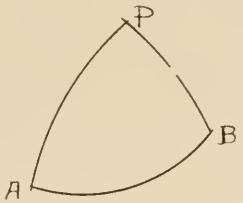

Fig. 1

locus of P was shown to be a small circle. This result was derived by Johann Lexell (1740-1784), an astronomer at St. Petersburg, in 1781. The problem was found to have been solved earlier, 1778, by Euler.[1] The result is sometimes known as Lexell's theorem.

A second spherical locus appeared as the solution of the problem: *To find the locus of the vertex of a spherical triangle upon a fixed base, such that the sum of the two variable sides is a constant.* This problem defines a locus

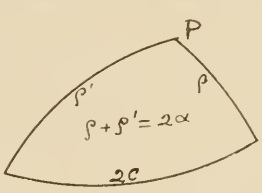

Fig. 2

upon the sphere analogous to the ordinary definition of an ellipse in the plane. The locus of P is called the *Spherical Ellipse.* The solution of this problem was found in 1785 by Nicholaus Fuss (1755-1826), a native of Basel, and an assistant to Euler at St. Petersburg from 1773 until Euler's death in 1783.

Frederick Theodore Schubert, a Russian astronomer, a contemporary of Fuss, published solutions to a number of spherical loci, types of which

[1] Cantor, Vol. IV, p. 384, p. 416.

are shown in the following: Given a triangle with a fixed base, find the locus of the vertex P such that the variable sides, ρ, ρ', Fig. 2, satisfy:

(1) $\sin\rho = k \sin\rho'$,
(2) $\cos\rho = k \cos\rho'$,
(3) $\sin\frac{\rho}{2} = k \sin\frac{\rho'}{2}$,
(4) $\cos\frac{\rho}{2} = k \cos\frac{\rho'}{2}$.

In Crelle's Journal, Vol. VI, 1830, pp. 244-254, Gudermann published an article "*Ueber die analytische Spharik*," which contains a collection of spherical loci connected with *sphero-conics*, for example, such as: (1) *The locus of the feet of perpendiculars drawn from the focus of a spherical ellipse upon tangents to the spherical ellipse*; (2) *The locus of the intersection of perpendicular tangents to a spherical ellipse*; and other problems similar to those of plane analytics. The notation employed by Gudermann is not fully explained, and is an adaptation from that used by him in a private publication of his work "*Grundriss der analytischen Spharik*, to which the present writer does not have access.

Thomas Stephens Davies published, 1834, in the Transactions of the Royal Society of Edinburgh, Vol. XII, pp. 259-362, and pp. 379-428, two papers, entitled, "*The Equations of Loci Traced upon the Surface of a Sphere*." In these extensive papers the author uses a system of polar coördinates upon the sphere, and derives the equations of many interesting curves, the spherical conics, cycloids, spirals, as well as many properties of these curves. The polar equations of Davies may be transformed into *great-circle* coördinates, giving equations of spherical loci in a form similar to the Cartesian equations of corresponding loci in the plane.

Spherical Analytics.

A system of analytic geometry upon the sphere may be derived in direct correspondence to that of the plane by a proper choice of axes of coördinates.

1. *Coördinates.* Let us select as axes two great circles XX′, YY′ perpendicular to each other at O, Fig. 3. The spherical coördinates of any point P are the intercepts, OA = ξ and OB = η, cut off upon the axes by perpendiculars drawn from P. Let the length of the perpendiculars from P be PB = ξ', and PA = η'.

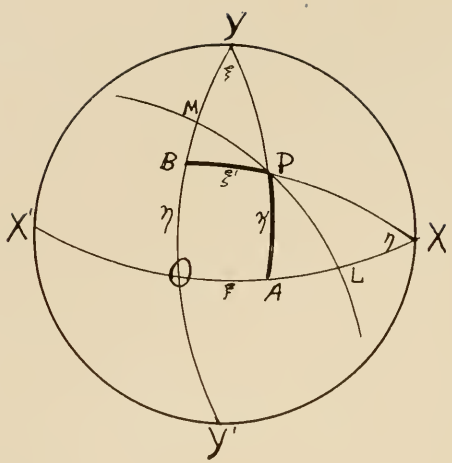

Fig. 3

From the right spherical triangles PBY and PAX we have the following fundamental relations:

$$(1)\quad \tan\xi = \frac{\tan\xi'}{\sin BY} = \frac{\tan\xi'}{\cos\eta},\quad \tan\eta = \frac{\tan\eta'}{\sin AX} = \frac{\tan\eta'}{\cos\xi}$$

2. *Equation of the Spheric Line LM in Terms of its Intercepts.*

The arc of a great circle we will call a *spheric straight line.* Let the intercepts be $OL = \alpha$, $OM = \beta$, and the angle $OLM = \phi$, Fig. 3. Then from the right triangles MOL and PAL we have

$$\tan\varphi = \frac{\tan\beta}{\sin\alpha},\ \text{and}\ \tan\varphi = \frac{\tan\eta'}{\sin AL} = \frac{\tan\eta'}{\sin(\alpha-\xi)}$$

Equating these values of $\tan\varphi$, and substituting the value of $\tan\eta'$ from (1),

$$\frac{\tan\beta}{\sin\alpha} = \frac{\tan\eta\cos\xi}{\sin\alpha\cos\xi - \cos\alpha\sin\xi} = \frac{\tan\eta}{\sin\alpha - \cos\alpha\tan\xi}$$

Expressing each function in terms of tangents and reducing, we find the equation of the spheric line in the intercept form:

$$(2)\quad \frac{\tan\xi}{\tan\alpha} + \frac{\tan\eta}{\tan\beta} = 1.$$

(1) *Special Cases.* (a) Parallels to the axes. A spheric line parallel to the OY-axis passes through the pole of the axis OX. Hence for a parallel to the OY-axis $\beta = 90°$ and the equation of the line becomes

$$\tan \xi = \tan \alpha \tag{3}$$

and for a parallel to the OX-axis, $\alpha = 90°$, and

$$\tan \xi = \tan \beta \tag{4}$$

(b) A line through one point. If a line (2) is to pass through (ξ_1, η_1), we have

$$\frac{\tan \xi - \tan \xi_1}{\tan \alpha} + \frac{\tan \eta - \tan \eta_1}{\tan \beta} = 0. \tag{5}$$

(c) A line through two points (ξ_1, η_1), (ξ_2, η_2), is given by

$$\frac{\tan \xi - \tan \xi_1}{\tan \xi_2 - \tan \xi_1} = \frac{\tan \eta - \tan \eta_1}{\tan \eta_2 - \tan \eta_1}$$

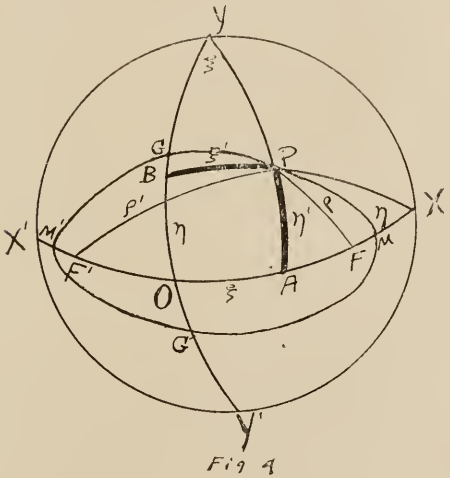

Fig 4

Conditions of *perpendicularity, parallelism, angles of intersection* of spheric straight lines may also be expressed, but will not be included here.

(2) *Correspondence to plane geometry.* The intercept form of the spheric straight line is similar to the corresponding equation in plane geometry, and may be reduced to that form by letting the radius of the sphere increase without limit.

3. *The Spheric Ellipse. Find the locus of the vertex P of a spherical triangle with fixed base FF′, such that the sum of the sides is a constant,* $\rho + \rho' = 2\alpha$. Fig. 4.

This definition defines the Spheric Ellipse MGM^1G^1.

Take the origin at the center O of the base FF′. Let $FF' = 2c$, $\rho + \rho' = 2\alpha$, $OM = \alpha$, $OG = \beta$. When P falls at G, $FG = \alpha = F'G$.

Then from the right triangle FOG (hypotenuse not drawn), we have

$$(1) \quad \cos\alpha = \cos\beta \cos c;$$

and from PAX,

$$(2) \quad \tan\eta' = \cos\xi \tan\eta.$$

From the right triangles PAF and PAF′, we have

(3) $\cos\rho = \cos\eta' \cos(c - \xi)$, $\cos\rho' = \cos\eta' \cos(c + \xi)$.

Adding equations (3) and using $\rho + \rho' = 2\alpha$,

$$(4) \quad \cos\alpha \cos\frac{\rho - \rho'}{2} = \cos\eta' \cos c \cos\xi.$$

and subtracting (3),

$$(5) \quad \sin\alpha \sin\frac{\rho - \rho'}{2} = \cos\eta' \sin c \sin\xi$$

Eliminating $\frac{\rho - \rho'}{2}$ and c from (1), (4), (5) and reducing, we find the symmetrical equation of the spheric ellipse

$$\frac{\tan^2\xi}{\tan^2\alpha} + \frac{\tan^2\eta}{\tan^2\beta} = 1,$$

α, and β being the intercepts on the axes, OM, and OG, respectively.

Special Cases. (1) Let $\alpha = \beta$, and we have a circle

$$(A) \quad \tan^2\xi + \tan^2\eta = \tan^2\alpha,$$

with center at O and radius α. With $\alpha = 90°$, this circle becomes the boundary of the hemisphere on which our geometry is located, corresponding to the circle with infinite radius in plane geometry.

(2) Let $\alpha = 90°$, and the ellipse becomes the two "*parallel lines*", $\tan^2\eta = \tan^2\beta$, passing through the poles of the OY-axis.

(3) The equation of a circle upon a sphere may be derived quite readily, but the resulting equation is somewhat unsymmetrical. Let ξ_1, η_1, be the

coördinates of the center, and let α be the radius. Then the equation may be derived from the fundamental equations

$$\tan \eta_1' = \cos \xi_1 \tan \eta_1, \tan \xi_1' = \cos \eta_1 \tan \xi_1,$$
$$\tan \eta' = \cos \xi \tan \eta, \tan \xi' = \cos \eta \tan \xi,$$

and the polar equation

$$\cos \alpha = \sin \eta_1' \sin \eta' + \cos \eta_1' \cos \eta' \cos (\xi - \xi_1),$$

by the elimination of ξ_1', η_1' and ξ', η'.

The resulting equation is

$$(\tan \xi - \tan \xi_1)^2 + (\tan \eta - \tan \eta_1)^2 + (\tan \xi \tan \eta_1 - \tan \xi_1 \tan \eta)^2 = \tan^2 \alpha (1 + \tan \xi \tan \xi_1 + \tan \eta \tan \eta_1)^2.$$

When $\xi_1 = \eta_1 = 0$, this equation reduces to that given in (A) above.

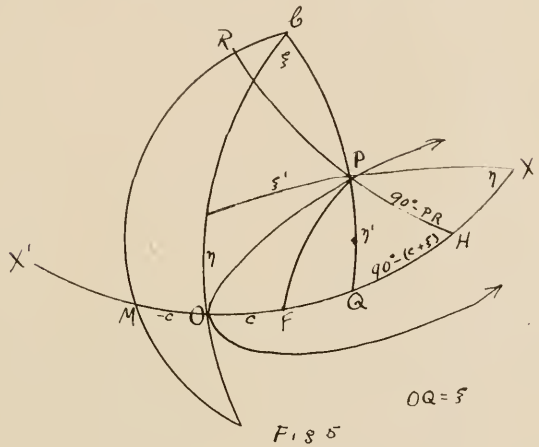

Fig 5

4. *The Spheric Hyperbola.* This spherical curve may be defined as *the locus of a point which moves so that the difference of its distances from two fixed points is constant,* $\rho - \rho' = 2\alpha$.

Using the notation of Fig. 4, but with $\rho - \rho' = 2\alpha$, this definition leads to the equation

$$\frac{\tan^2 \xi}{\tan^2 \alpha} - \frac{\tan^2 \eta}{\tan^2 \beta} = 1.$$

which is the spheric hyperbola. The locus does not intersect the OY-axis; the conjugate spheric hyperbola may be defined by

$$\frac{\tan^2 \xi}{\tan^2 \alpha} - \frac{\tan^2 \eta}{\tan^2 \beta} = -1,$$

and the spheric *asymptotes* to either by

$$\frac{\tan \xi}{\tan \alpha} = \pm \frac{\tan \eta}{\tan \beta}$$

5. *The Spheric Parabola.* A Spheric Parabola may be defined as *the locus of a point moving upon the surface of a sphere so as to be equally distant from a fixed point F and a fixed great circle CM, Fig.* 5.

From the definition PR = PF; let O bisect MF. Then from Fig. 5,

(1) $\tan \eta' = \cos \xi \tan \eta$,

(2) $\cos PH = \sin PR = \cos \eta' \sin (c + \xi)$,

(3) $\cos PF = \cos \eta' \cos (\xi - c)$.

Squaring and adding (2), (3)

$$1 = \cos^2 \eta' \sin^2 (\xi + c) + \cos^2 (\xi - c),$$

or

$$1 + \tan^2 \eta' = 1 + 4 \sin c \cos c \sin \xi \cos \xi.$$

Substituting from (1),

$$\tan^2 \eta = 2 \sin 2c \tan \xi,$$

which is the required equation.

6. *Correspondence to Plane Geometry.* The above equations of the *spheric straight line, ellipse, hyperbola, parabola, and circle*, show a marked similarity to the corresponding equations in the plane. These equations may be reduced to the equations in plano by considering the radius of the sphere to increase without limit. This may be done by expressing the arcs in terms of the radius, and finding the limit of the functions in each equation as $r \to \infty$.

For example, in the spheric ellipse,

$$(1) \quad \frac{\tan^2 \xi}{\tan^2 \alpha} + \frac{\tan^2 \eta}{\tan^2 \beta} = 1,$$

let (ξ, η), (α, β) be radian measure of arcs on a unit sphere; then on a sphere of radius r, we have arcs (x, y), (a, b) determined by

$$\xi = \frac{x}{r}, \quad \eta = \frac{y}{r}, \quad \alpha = \frac{a}{r}, \quad \beta = \frac{b}{r}.$$

Equation (1) becomes

$$\frac{\tan^2\left\{\frac{x}{r}\right\}}{\tan^2\left\{\frac{a}{r}\right\}} + \frac{\tan^2\left\{\frac{y}{r}\right\}}{\tan^2\left\{\frac{b}{r}\right\}} = 1.$$

Expand the tangents into infinite series according to the law

$$\tan Z = Z + \frac{Z^3}{3} + \frac{2Z^5}{15} + \frac{17\ Z\ \text{“7” exponent of Z,}}{315} + \frac{\quad}{\quad}$$

and we find

$$\frac{\left\{\frac{x}{r} + \frac{x^3}{3r^3} \ldots\right\}^2}{\left\{\frac{a}{r} + \frac{a^3}{3r^3} \ldots\right\}^2} \quad \frac{\left\{\frac{y}{r} + \frac{y^3}{3r^3} + \ldots\right\}^2}{\left\{\frac{b}{r} + \frac{b^3}{3r^3} + \ldots\right\}^2} = 1.$$

Dividing r^2 from each fraction, and passing to the limit $r = \infty$, and we have the equation of an ellipse in the plane,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Any equation in the “rectangular spheric” coördinates will reduce, in the limit when the sphere is made to increase infinitely, to the equation of a corresponding locus in the plane.