Irrelevant Factors in Bitangentials of Plane Algebraic Curves.

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Three years ago I presented a paper to the mathematical section of the Academy dealing with the proof of a formula used by Mr. Heal in an article published in the Annals of Mathematics, vol. VI, page 64. This formula was used by Heal in freeing a bitangential of the plane quintic, which he had developed in a previous paper in the Annals, vol. V, page 33, from an irrelevant factor, the square of the hessian of the quintic. Since then I have continued the study of the subject and wish to present an interesting result in the light of Heal's work.

Taking the general equation in the symbolic notation

$$(a_1 x_1 + a_2 x_2 + a_3 x_3)^n = a_x^n = b_x^n = c_x^n = \cdots = 0, \quad \ldots\ldots (1)$$

for the n-ic and deriving the first polar, with respect to the n-ic, of any point y, we have

$$(a_1 x_1 + a_2 x_2 + a_3 x_3)^{n-1} (a_1 y_1 + a_2 y_2 + a_3 y_3) = a_x^{n-1} a_y = 0, \ldots . (2)$$

Any point on the line through the points x and y may be represented by $\lambda x + \mu y$, where λ and μ have a fixed ratio for any particular point. If x be a point on the n-ic and y be a point on the tangent to the n-ic at the point x, then we have equations (1) and (2) satisfied by the points x and y respectively, and equation (2), as an equation in y, represents the tangent to the n-ic at x. If, in addition to these conditions, the point $\lambda x + \mu y$ lie on the n-ic, we must have from (1)

$$\left\{ a_{\lambda x + \mu y} \right\}^n = (\lambda a_x + \mu a_y)^n = 0,$$

from which, by virtue of (1) and (2), we get

$$\frac{n(n-1)}{2!} a_x^{n-2} a_y^2 \lambda^{n-2} + \frac{n(n-1)(n-2)}{3!} a_x^{n-3} a_y^3 \lambda^{n-3} \mu + \cdots +$$

$$n a_x a_y^{n-1} \lambda \mu^{n-3} + a_y^n \mu^{n-2} = 0. \quad \ldots \quad (3)$$

Equation (3) is an (n-2)-ic in λ and μ which gives the positions of the remaining n-2 intersections of the tangent to the n-ic at x with the n-ic itself. In order that this tangent be a bitangent the discriminant of equation (3) must vanish. This discriminant is a function of x and y, and if y

can be expressed in terms of x, then the discriminant becomes a bitangential of the n-ic. It has been shown by Jacobi and Clebsch that this is always possible.

We shall write equation (3) as

$$A_0\,\lambda^{n-2} + (n-2)\,A_1\,\lambda^{n-3}\,\mu + \frac{(n-2)\,(n-3)}{2\,!}\,A_2\,\lambda^{n-4}\,\mu^2 + \cdots +$$
$$(n-2)\,A_{n-3}\,\lambda\mu^{n-3} + A_{n-2}\,\mu^{n-2} = 0, \ \ldots\ (4)$$

where we have

$$A_0 = \frac{n\,(n-1)}{1\,.\,2}\,a_x^{n-2}\,a_y^2,\ A_1 = \frac{n\,(n-1)}{2\,.\,3}\,a_x^{n-3}\,a_y^3,\ \cdots\cdots$$
$$A_r = \frac{n\,(n-1)}{(r+1)\,(r+2)}\,a_x^{n-r-2}\,a_y^{r+2}.$$

If equation (4) is a quadratic, that is, if the n-ic is a quartic, the discriminant of (4) is

$$-\frac{4}{A_0^2}\,(A_0\,A_2 - A_1^2) = 0,$$

and after y is expressed in terms of x there is no irrelevant factor.

If the n-ic be the quintic, the discriminant of (4) is

$$-\frac{27}{A_0^6}\,(G^2 + 4\,H^3) = 0,$$

where we put $H = A_0\,A_2 - A_1^2$ and $G = A_0^2\,A_3 - 3\,A_0\,A_1\,A_2 + 2\,A_1^3$, and the y can easily be expressed in terms of x for the functions G and H, but the result contains the square of the hessian of the quintic as an irrelevant factor. This factor can be discarded without difficulty by putting

$$G^2 + 4\,H^3 = A_0^2\left\{(A_0\,A_3 - A_1\,A_2)^2 - 4\,(A_0\,A_2 - A_1^2)\,(A_1\,A_3 - A_2^2)\right\},$$

and then expressing y in terms of x for each parenthesis separately.

If the n-ic be the sextic, the discriminant of (4) is

$$\frac{256}{A_0^6}\,(I^3 - 27J^2) = 0,$$

where $I = A_0\,A_4 - 4\,A\,A_1,\ A_3 + 3A_2^2$ and $A_0^3\,J = A_0\,H\,I - G^2 - 4H^3$.

There is no difficulty in expressing y in terms of x for the function I, and therefore, by multiplying and dividing the discriminant by A_0^6, we can immediately write a bitangential of the sextic by substituting the results obtained for the quartic and quintic in

$$\frac{256}{A_0^{12}}\left\{A_0^6\,I^3 - 27\,(A_0\,H\,I - G^2 - 4\,H^3)\right\} = 0.$$

But this bitangential of the sextic contains the sixth power of the hessian of the sextic as an irrelevant factor. In order to free it from this factor, we put

$$J = (A_0 A_2 - A_1^2) A_4 - (A_0 A_3 - A_1 A_2) A_3 + (A_1 A_3 - A_2^2) A_2,$$

and then express y in terms of x for the function J. The work involved in this last step is very long and tedious. These results can be used in developing a bitangential of the septic, but two additional functions will have to be developed, the work in which is almost beyond the range of possibility.