Geodesic Lines on the Syntractrix of Revolution.

E. L. Hancock.

The syntractrix is defined as a curve formed by taking a constant length, d upon the tangent c to the tractrix*. The surface formed by revolving this curve about its asymptote is the one under consideration. We shall call it S.

Being a surface of revolution it is represented by the equations

$$x = u \cos v$$
$$y = u \sin v$$
$$z = -\sqrt{d^2 - u^2} + \frac{c}{2} \log \frac{d + \sqrt{d^2 - u^2}}{d - \sqrt{d^2 - u^2}}$$

Using the Gaussian notation† we find:

$$E = \frac{u^2(d^2 - 2cd) + c^2d^2}{u^2(d^2 - u^2)},\ F = o,\ G = u^2,\ A = -\frac{u^2 - cd}{u\sqrt{d^2 - u^2}} u \cos v,\ B = -\frac{u^2 - cd}{u\sqrt{d^2 - u^2}} u \sin v\ C = u,\ D = \frac{u^2(d^2 - 2cd) + cd^3}{u(d^2 - u^2)^{\frac{3}{2}}},\ D' = o,\ D'' = \frac{u(u^2 - cd)}{\sqrt{d^2 - u^2}}$$

$$K = \frac{1}{R_1R_2} = \frac{DD'' - D'^2}{EG - F^2} = \frac{(u^2 - cd)[u^2(d - 2c) + cd^2]}{(d^2 - u^2)[u^2(d - 2c) + c^2d}$$

In the particular surface given by $d = 2c$ the Gaussian curvature becomes

$$\frac{2\left(u^2 - \frac{d^2}{2}\right)}{d^2 - u^2}$$

Here d is positive, and since $d > u$, the denominator is always positive. We get the character of the curvature of different parts of the surface by considering the numerator. When $u^2 = d^2/2$, $K = O$, i. e., the circle $u = d/2$ $\frac{d}{\sqrt{2}}$ is made up of points having zero-curvature. When $u^2 > d^2/2$, $K > O$, and when $u^2 < d^2/2$, $K < O$.

For this particular surface

$$E = \frac{d^4}{4u^2(d^2 - u^2)},\ F = o,\ G = u^2,\ A = -\frac{2u^2 - d^2}{2u\sqrt{d^2 - u^2}} u \cos v,\ B = -\frac{2u^2 - d^2}{2u\sqrt{d^2 - u^2}} u \sin v\ C = o,\ D = \frac{d^4}{2u(d^2 - u^2)^{\frac{3}{2}}},\ D' = o,\ D'' = \frac{u(2u^2 - d^2}{2\sqrt{d^2 - u^2}}$$

To get the geodesic lines of the surface we make use of the method of the calculus of variations according Weierstrass§. This requires that we minimize the integral:

---

* Peacock, p. 175.

† Bianchi, Differential Geometrie, pp. 61, 87, 105.

§ Osgood, Annals of Mathematics, Vol. II (1901), p. 105.

$$I = \int_{t^2}^{t^1} \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}.dt$$

Denote $\sqrt{E\,u'^2 + 2\,F\,u'\,v' + G\,v'^2}$ by F. Then the first condition for a minimum of I is $Fv - \frac{d}{dt}Fv' = o$||

Now, in this case $Fv = o$, so that $\frac{d}{dt}Fv' = o$

Hence $Fv' = \delta$, or substituting the values E, F and G this becomes

$$\frac{u^2\,v^1}{\sqrt{\frac{d^4\,u'^2}{4\,u^2(d^2 - u_2)} + u^2\,v'^2}} = \delta$$

When $\delta = o$, $v' = o$, hence $v = \text{constant}$, i. e., the meridians are geodesic lines.

When $\delta = o$

$$(1)\quad v = \int \frac{\delta\,d^2\,u^1}{2\,u^2\sqrt{(d^2 - u^2)\,(u^2 - \delta^2)}} + \delta'$$

Making the substitution $u = 1\ t$, (1) becomes

$$(2)\quad v = \int \frac{-\delta\,d^2\,t^2\,dt}{2\sqrt{(t^2\,d^2 - 1)\,(1 - \delta^2\,t^2)}} + \delta'$$

We have for the reduction of the general elliptic integral

$$*R(x) = A\,x^4 + 4\,B\,x^3 + 6\,C\,x^2 + 4\,B'\,x + A'$$

$$g_2 = AA' - 4\,BB' + 3\,C^2$$

$$g_3 = AcA' + 2\,BcB' - A'B^2 - AB'^2 - c^3.$$

These become in the present case

$$R(t) = (t^2d^2 - 1)(1 - \delta^2t^2) = -\delta^2d^2t^4 + (d^2 + \delta^2)t^2 - 1$$

$$g_2 = \delta^2d^2 + \frac{(d^2 + \delta^2)}{12}$$

$$g_3 = \frac{\delta^2\,d^2(d^2 + \delta^2)}{6} - \left(\frac{d^2 + \delta^2}{6}\right)^3$$

We get also

$$R'(t) = -4\,\delta^2d^2t^3 + 2\,(d^2 + \delta^2)\,t$$

$$R'(t) = -12\,\delta^2d^2t^2 + 2\,(d^2 + \delta^2)$$

Making the substitution

$$(3)\quad t = a + \frac{\frac{1}{4}\,R'(a)}{p\,u - \frac{1}{24}\,R''(a)}\dagger$$

Where a is one of the roots of R(t), say 1 d, we get

$$t = \frac{1}{d} + \frac{\frac{\frac{1}{2}(d^2 - \delta^2)}{d}}{p\,u - p\,v} \quad \text{where } p\,v = \tfrac{1}{12}(d^2 - 5\delta^2)$$

|| Kneser, Variationsrechnung. Fv denotes function v.

* Klein, Ellip. Mod. Functionen, Vol. I, p. 15.

† Enneper, Ellip. Functionen, 1890, p. 30.

Now, since $\frac{dt}{du} = \sqrt{R(t)}$ we get from (2)

$$(4)\quad v = -\frac{\delta d}{2}\int t^2 du + \delta' = \frac{1}{2\delta}\int \left[-\delta^2 - \frac{\delta^2(d^2-\delta^2)}{p\,u - p\,v} - \frac{\frac{\delta^2}{4}(d^2-\delta^2)^2}{(p\,u - p\,v)^2} \right] du + \delta'$$

Noting that in the present case

$$(p'\,v)^2 = -\frac{\delta^2}{4}(d^2-\delta^2)^2$$

$$p''\,v = -\delta^2(d^2-\delta^2)$$

and remembering that

$$\frac{(p'\,v)^2}{p\,u - p\,v)^2} = p(u+v) - p)u - v) - 2\,p\,v - \frac{p''\,v}{p\,u - p\,v}$$

(4) becomes

$$v = \frac{1}{2\delta}\int [-\delta^2 + p(u+v) - p(u-v) - 2\,p\,v]du + \delta'$$

$$= \frac{1}{2\delta}\left[-\frac{1}{6}(d^2+\delta^2)u + \frac{\sigma'}{\sigma}(u-v) - \frac{\sigma'}{\sigma}(u+v) \right] + \delta'$$

The functions $\frac{\sigma'}{\sigma}$ may be expressed in power series. We have then the geodesic lines given by the equations

$$v = f(t) + \delta'$$

$$u = \frac{1}{t}$$

The constant δ' being additive has no effect upon the nature of the geodesics. It determines their position. All lines given by δ' may be made to coincide by a revolution about the z-axis. The curves may be completely discussed when $\delta' = o$.

Since the parameter lines of the surface consist of geodesic lines through a point and their orthogonal trajectories E may be taken equal to unity.* $E\,du^2 = du'^2$

Hence $-\frac{d}{2}\log\left\{\frac{d+\sqrt{d^2-u^2}}{u}\right\} = u'$, or $u = d\,\text{sech}\,\frac{2u'}{d}$

Because of the relations of the surface to the pseudo-sphere it may be represented upon the upper part of the Cartesian plane†. The relation between the surfaces is given by the equations

$$v = v'$$

$$u = \frac{c}{d}u'$$

---

* Knoblauch, Krummen Flächen, p. 49.

† Bianchi, Differential Geometrie, p. 419.

where u and v are co-ordinates of points on the pseudo-sphere and u' and v' co-ordinates of points on S. The equations of transformation from S to the plane are

$$v = x$$

$$c\, e^{-\frac{u}{d}} = y$$

The real part of the surface being represented on the strip included between $y = c$ and $y = c\, e$.

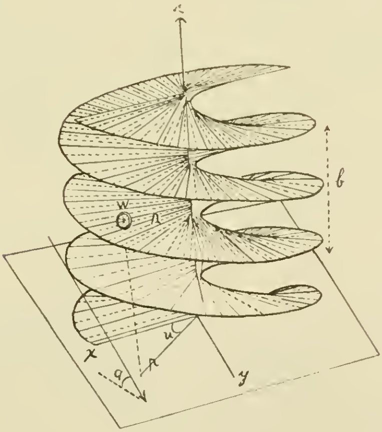
Comparison of Gauss' and Cayley's Proofs of the Existence Theorem.

O. E. Glenn.

[By title.]

Motion of a Bicycle on a Helix Track.

O. E. Glenn.



The equation of the helix surface may be conveniently expressed in surface co-ordinates, thus:

$$x = r \cos u \equiv f_1 (ru)$$

$$y = r \sin u \equiv f_2 (ru)$$

$$z = \frac{bu}{2\pi} \equiv f_3 (ru)$$

in which r represents the distance of a point from the z axis, and u the