CHECKS ON COMPUTATIONS IN THE SOLUTION OF TRIANGLES

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It is the purpose of this note to illustrate methods of checking the accuracy of the results when unknown parts of plane triangles are computed from given parts. Five place tables are used in the computations.

I. RIGHT TRIANGLES.

Let h represent the hypotenuse and a and b the other two sides of a right triangle Let R be the right angle and A and B the acute angles opposite a and b respectively. To fix ideas suppose A is not less than B.

Either of the following identities contains all five of the variable parts and can be used as a check formula when a right triangle has been completely solved.

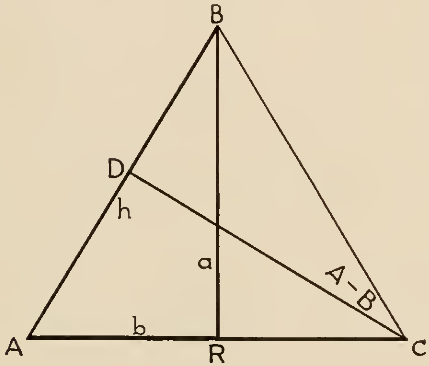

Fig. 1

$$(1) \qquad 2\,ab = h^2 \cos\ (A-B)$$

$$(2) \qquad (a+b)\ (a-b) = h^2 \sin\ (A-B)$$

To prove these produce AR (Fig. 1) to C making $RC = AR$, connect BC, and draw CD perpendicular to AB. Then $CB = h$ and angle $BCD = A-B$.

$$CD = h \cos\ (A-B) = 2b \sin\ A$$

Therefore $$h^2 \cos\ (A-B) = 2bh \sin\ A = 2ab$$

$$DB = h \sin\ (A-B) = h - 2b \cos\ A$$

Therefore $$h^2 \sin\ (A-B) = h^2 - 2bh \cos\ A = a^2 - b^2$$

It is evident that these formulas hold also when A is less than B.
If it is desired to check only the sides, either of the formulas

$$(3) \quad a = (h+b)\ (h-b) \text{ or } b^2 = (h+a)\ (h-a)$$

may be used.

Example 1. Given $A = 63°$, $h = 28.54$. Compute $B = 27°$, $\log a = 1.40533$, $a = 25.429$, $\log b = 1.11250$, $b = 12.957$.

CHECKS

$A = 63°$		$2ab = h^2 \cos (A-B)$
$B = 27°$	$\log 2 = 0.30103$	$\log h = 1.45545$
	$\log a = 1.40533$	1.45545
$A-B = 36$	$\log b = 1.11250$	$\log \cos (A-B) = 9.90796-10$
	2.81886	2.81886
$h = 28.54$		
	$(a+b)(a-b) = h^2 \sin (A-B)$	
$a = 25.429$	1.58417	1.45545
$b = 12.957$	1.09594	1.45545
		9.76922−10
$a+b = 38.386$	2.68011	2.68012
$a-b = 12.472$	$a^2 = (h+b)(h-b)$	
$h+a = 53.969$	1.40533	1.61802
	2	1.19265
$h-a = 3.111$	2.81066	2.81067
$h+b = 41.497$	$b^2 = (h+a)(h-a)$	
$h-b = 15.583$	1.11250	1.73214
	2	0.49290
	2.22500	2.22504

Example 2. Given $A = 28°\ 40'.4$, $b = 20.71$ Compute $B = 61°19'.6$ $\log a = 1.057407$, $a = 11.326$, $\log h = 1.37300$, $h = 23.605$.

CHECKS

$B = 61°\ 19'.6$	$2ab = h^2 \cos (B-A)$	
$A = 28°\ 40'.4$	0.30103	1.37300
	1.05407	1.37300
$B-A = 32°\ 39'.2$	1.31618	9.92528−10
	2.67128	2.67128
$h = 23.605$		
	$(b+a)(b-a) = h^2 \sin (B-A)$	
$b = 20.71$		1.37300
	1.50564	1.37300
$a = 11.326$	0.97239	9.73204
	2.47803	2.47804
$b+a = 32.036$		
	$a^2 = (h+b)(h-b)$	
$b-a = 9.384$		
	1.05407	1.64655
$h+a = 34.931$	2	0.46165
	2.10814	2.10820
$h-a = 12.279$		
	$b^2 = (h+a)(h-a)$	
$h+b = 44.315$	1.31618	1.54321
	2	1.08916
$h-b = 2.895$		
	2.63236	2.63237

It appears that these checks are not all sensitive to the same degree. Experience will assist the computer in choosing the one best adapted to the problem at hand. For example, (1) is more sensitive than (2) when the difference of the angles is less than 45° and vice versa. Of (3) that one is better in which the factors are most nearly equal.

II. OBLIQUE TRIANGLES

When any triangle has been completely solved the formulas

$$(4) \qquad (a-b) \cos \tfrac{1}{2}C = c \sin \tfrac{1}{2}(A-B)$$

$$(5) \qquad (a+b) \sin \tfrac{1}{2}C = c \cos \tfrac{1}{2}(A-B)$$

$$(6) \qquad (a-b)(a+b) \sin C = c^2 \sin (A-B)$$

together with those obtained from these by cyclic permutations of the letters representing the sides and angles, may be used as checks.

Formulas (4) and (5) may be proved as follows and (6) is readily deduced from them.

Let ABC be any triangle having two sides unequal, say $a>b$. With a radius b, the shorter of the two unequal sides, and centre C, the vertex of their included angle, describe a circle through A which cuts the side CB in a point D between B and C and also at a second point E beyond C. Draw EA and at B erect a perpendicular which meets EA produced in F. On DF as diameter construct a circle; this circle will pass through A and B. Then angle $BEF = \frac{1}{2}C$, $DFA = B$, $BFE = \frac{1}{2}(A+B)$, and $BFD = \frac{1}{2}(A-B)$.

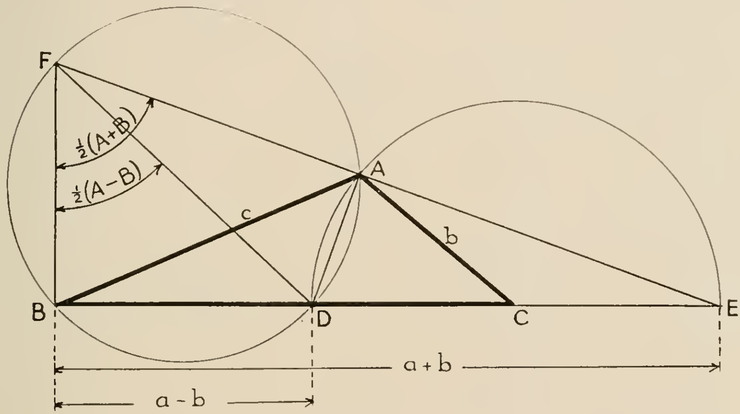

Fig. 2

In the triangle ABD,

$$\frac{a-b}{c} = \frac{\sin BAD}{\sin BDA}$$

but $\sin BAD = \sin BFD = \sin \frac{1}{2}(A-B)$

and $\sin BDA = \sin ADE = \cos AED = \cos \frac{1}{2}C$

Therefore

$$(a-b) \cos \tfrac{1}{2}C = c \sin \tfrac{1}{2}(A-B)$$

In the triangle ABE,

$$\frac{a+b}{c} = \frac{\sin BAE}{\sin AEB}$$

but $\sin BAE = \sin BAF = \sin BDF = \cos BFD = \cos \frac{1}{2}(A-B)$

and $\sin AEB = \sin \frac{1}{2}C$

Therefore

$$(a+b) \sin \tfrac{1}{2}C = c \cos \tfrac{1}{2}(A-B)$$

Case 1. Given a side and two angles.

Example 1. Given $a = 2.903$, $B = 79° 40'$, $C = 33° 15'$. Compute $A = 67° 5'$ and $\log b = 0.49146$, $b = 3.1007$, $\log c = 0.23757$, $c = 1.7281$, by the law of sines.

$b = 3.1007$
$c = 1.7281$

$b + c = 4.8288$

$b - c = 1.3726$
$B = 79° 40'$
$C = 33° 15'$

$B - C = 46° 25'$

$\tfrac{1}{2}(B - C) = 23° 12'.5$

$\tfrac{1}{2}A = 33° 32'.5$

CHECKS

$(b - c) \cos \tfrac{1}{2}A = a \sin \tfrac{1}{2}(B - C)$

0.13754	0.46285
9.92090–10	9.59558–10
0.05844	0.05843

$(b + c) \sin \tfrac{1}{2}A = a \cos \tfrac{1}{2}(B - C)$

0.68384	0.46285
9.74236–10	9.96335–10
0.42610	0.42610

$(b + c)(b - c) \sin A = a^2 \sin (B - C)$

0.68384	0.46285
0.13754	0.46285
9.96429–10	9.85996–10
0.78567	0.78566

Case 2. Given two sides and their included angle.

Example 1. $a = 22$, $b = 12$, $C = 42°$. Compute $c = 15.350$ by the law of cosines.

$a = 22$
$b = 12$
$c = 15.350$

$2s = 49.350$
$s = 24.675$

$s - a = 2.675$
$s - b = 12.675$
$s - c = 9.325$

Check $= 24.675$

CHECK

$s(s - c) \tan^2 \tfrac{1}{2}C = (s - a)(s - b)$

1.39226	
0.96965	
9.58418–10	0.42732
9.58418–10	1.10295
1.53027	1.53027

Compute $A = 106° 27'.7$ and $B = 31° 32'.4$ by law of sines.

$a + b = 34$
$a - b = 10$

$c = 15.35$

$A = 106° 27'.7$

$B = 31° 32'.4$

$A - B = 74° 53'.3$

$\tfrac{1}{2}(A - B) = 37° 27'.6$

$\tfrac{1}{2}C = 21°$

CHECKS

$(a - b) \cos \tfrac{1}{2}C = c \sin \tfrac{1}{2}(A - B)$

1.00000	1.18611
9.97015–10	9.78405–10
0.97015	0.97016

$(a + b) \sin \tfrac{1}{2}C = c \cos \tfrac{1}{2}(A - B)$

1.53148	1.18611
9.55433–10	9.89970–10
1.08581	1.08581

$(a + b)(a - b) \sin C = c^2 \sin (A - B)$

1.53148	1.18611
1.00000	1.18611
9.82551–10	9.98478–10
2.53699	2.53700

Example 2. Given $a = 34.645$, $b = 22.531$, $C = 43° 31'$.

$a = 34.645$
$b = 22.531$

$a + b = 57.176$
$a - b = 12.114$

$\tfrac{1}{2}C = 21° 45'.5$
Then compute $\tfrac{1}{2}(A + B) = 68° 14'.5$, $\tfrac{1}{2}(A - B) = 27° 57'.6$
whence $A = 96° 12'.1$
and $B = 40° 16'.9$

CHECK

$a \sin B = b \sin A$

1.53964	1.35278
9.81060–10	9.99745–10
1.35024	1.35023

Compute $\log c = 1.38014$, $c = 23.996$ by the law of sines, in two ways.

CHECKS

$(a-b) \cos \frac{1}{2}C = c \sin \frac{1}{2}(A-B)$ | $(a+b) \sin \frac{1}{2}C = c \cos \frac{1}{2}(A-B)$

1.08328	1.38014	1.75721	1.38014
9.96790–10	9.67104–10	9.56902–10	9.94610–10
1.05118	1.05118	1.32623	1.32624

$(a+b)(a-b) \sin C = c^2 \sin (A-B)$

1.75721	1.38014
1.08328	1.38014
9.83795–10	9.91817–10
2.67844	2.67845

Case 3. Given the three sides.

Example. Given $a = 2314$, $b = 2431$, $c = 3124$. Compute $\frac{1}{2}A = 23° \ 36'.8$, $\frac{1}{2}B = 25° \ 13'.8$, $\frac{1}{2}C = 41° \ 9'.4$ and check by taking their sum.

CHECKS

$a \sin B = b \sin A$ | $b \sin C = c \sin B$

3.36436	3.38578	3.38578	3.49471
9.88716–10	9.86572–10	9.99608–10	9.88716–10
3.25152	3.25150	3.38186	3.38187

$a \sin C = c \sin A$ | $(c-a) \cos \frac{1}{2}B = b \sin \frac{1}{2}(C-A)$

3.36436	3.49471	2.90849	3.38578
9.99608–10	9.86572–10	9.95646–10	9.47918–10
3.36044	3.36043	2.84495	2.84496

$c + a = 5438$

$c - a = 810$

$(c+a) \sin \frac{1}{2}B = b \cos \frac{1}{2}(C-A)$

3.73544	3.38578
9.62967–10	9.97932–10
3.36511	3.36510

$\frac{1}{2}(C-A) = 17° \ 32'.6$

$(C-A) = 35° \ 5'.2$

$(c+a)(c-a) \sin B = b^2 \sin (C-A)$

2.90849	3.38578
3.73544	3.38578
9.88716–10	9.75953–10
6.53109	6.53109

Case 4. Given two sides (say a and b) and the angle opposite one of them (say A).

Determine the number of solutions. Compute the angle B opposite the other given side, by the law of sines. If there are two solutions call the acute angle B_1, and the obtuse one B_2.

It is now possible to check by the law of tangents but this is in many cases not sensitive enough to be decisive. Find the third angle C by subtracting $A+B$ from 180°, and compute the third side c by the law of sines. If there are two solutions a check is given by the formulas,

(7) $B_1 = A + C_2, \quad c_1 + c_2 = 2b \cos A$

$B_2 = A + C_1, \quad c_1 - c_2 = 2a \cos B_2$

Example. Given $a = 301.35$, $b = 352.11$, $A = 33° 17'$.
There are two solutions. Compute log sin $B = 9.80701-10$, $B_1 = 39° 53'$, $B_2 = 140° 7'$. Check by the law of tangents.

$b+a = 653.46$	$(b+a) \tan \frac{1}{2}(B_1-A) =$	$(b-a) \tan \frac{1}{2}(B_1+A)$
$b-a = 50.76$	2.81522	1.70552
	8.76087−10	9.87053−10
$\frac{1}{2}(B_1+A) = 36° 35'$	1.57609	7.57605
	$(b+a) \tan \frac{1}{2}(B_2-A) =$	$(b-a) \tan \frac{1}{2}(B_2+A)$
$\frac{1}{2}(B_1-A) = 3° 18'$		
	2.81522	1.70552
$\frac{1}{2}(B_2+A) = 86° 42'$	0.12947	1.23913
	2.94469	2.94465
$\frac{1}{2}(B_2-A) = 53° 25'$		
Next compute	$C_1 = 106° 50'$,	$C_2 = 6° 36'$
	$A = 33° 17'$	$A = 33° 17'$
Check	$B_2 = 140° 7'$	$B_1 = 39° 53'$

Now compute c_1 and c_2 by the law of sines, log $c_1 = 2.72065$, $c_1 = 525.59$, log $c_2 = 1.80013$, $c_2 = 63.114$; whence $c_1+c_2 = 588.704$, $c_1-c_2 = 462.476$

CHECKS

$c_1+c_2 =$	$2b \cos A$	$c_1-c_2 =$	$2a \cos B$
	0.30103		0.30103
	2.54668		2.47907
	9.92219−10		9.88499−10
2.76989	2.76990	2.66509	2.66509

The triangles now being completely solved, any of the checks illustrated above may be used; for example

$(b+a) \sin \frac{1}{2}C_1 =$	$c \cos \frac{1}{2}(B_1-A)$
2.81522	2.72065
9.90471−10	9.99928−10
2.71993	2.71993

Purdue University, December 1, 1919