Note on "Note on Smith's Definition of Multiplication." By A. L. Baker.

The rule should be: To multiply one quantity by another, perform upon the multiplicand the series of operations which was performed upon unity to produce the multiplier.

This does not mean, perform upon the multiplicand the series of successive operations which was performed upon unity and upon the successive results.

Thus, to multiply b by $\sqrt{a}$: If we attempt to consider $\sqrt{a}$ as derived by taking unity a times and then extracting the square root of the result, we violate the rule. To get $\sqrt{a}$ by performing operations upon unity, we must (e. g., $a=2$) take unity 1 time, .4 times, .01 times, .004 times, etc., and add the results. Doing this to b, we get the correct result, viz., $\sqrt{2}\,b=1.414\ldots b$.

The rule is thus universal, applying to all multipliers, complex, quaternion and irrational.

The Geometry of Simson's Line. By C. E. Smith, Indiana University.

1. If from any point in the circumference of the circumcircle to a $\triangle$ ABC $\perp$s to the sides of the $\triangle$ be drawn, their feet, P_1, P_2, and P_3, lie in a straight line. This is known as Simson's Line.

(a) First proof that P_1, P_2, and P_3 lie in a straight line.

Since $\angle$s PP_3B and PP_1B (Fig. 1.) are both right $\angle$s, P, P_3, P_1 and B are concyclic.

Likewise P, P_2, A, and P_3 are concyclic.

Now $\angle PP_3P_1 + \angle PBP_1 = 180°$.

and $\angle PAC + \angle PBP_1 = 180°$.

$\therefore \angle PP_3P_1 = \angle PAC$,

But $\angle PAC + \angle PAP_2 = 180°$.

$\therefore \angle PP_3P_1 + \angle PAP_2 = 180°$.

But $\angle PAP_2 = \angle PP_3P_2$ (measured by same arc of auxiliary circle)

$\therefore \angle PP_3P_1 + \angle PP_3P_2 = 180°$, or a straight $\angle$.

$\therefore$ P_1 P_3 and P_2 lie in a straight line.

(b) Second proof that P_1, P_2, and P_3 lie in a straight line.

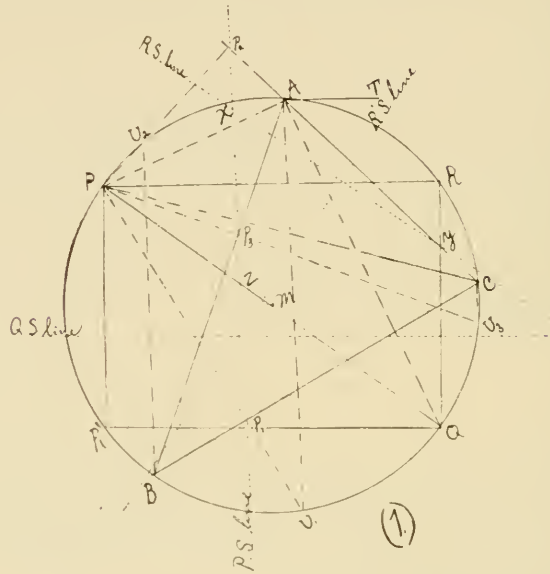
Draw PC and PA (Fig. 1).

Now $\angle$s PP_2C and PP_1C are right $\angle$s.

$\therefore$ P, P_1, C and P_2 are concyclic with PC as diameter.

$\angle PAB = \angle PCB = \angle PCP_1$,

and $\angle PP_2P_1 = \angle PCP_1$,



$\therefore \angle PAB = \angle PAP_3 = PP_2P_1$

Now P, P_2, A, P_3 are concyclic.

$\therefore \angle PP_2P_3 = PAP$ and

$\therefore \angle PP_2P_3 = \angle PP_2P_1$

$\therefore P_2P_1$ passes through P_3 and the three points are collinear.

2. If PP_1 be produced until it intersects the circumcircle of $\triangle$ ABC, at the point U_1, then AU_1 is $\parallel$ to Simson's line of P. (Fig. 1.)

Now the points P, P_1, P_2, and C are concyclic.

$\therefore$ the $\angle P_1PC = \angle P_1P_2C$.

But $\angle P_1PC = \angle U_1AC$, (arc CU_1 common to both)

and $\angle P_1P_2C = U_1AC$.

If two angles are equal and have a pair of sides in coincidence, then the other sides must also either coincide or be parallel. Hence $AU \parallel P_1 P_2 P_3$, or to Simson's line. Thus we can show BU_2 and CU_3 parallel to Simson's line of P and therefore AU_1, BU_2 and CU_3 are parallel to each other.

3. Let AT (Fig. 1) be isogonal conjugate to AP. Then Simson's line of $P \perp AT$.

Also Simson's line of $T \perp AP$.

Now, AU_1 is $\parallel$ Simson's line of P, and

$\angle BAT = \angle PAC = 180° - \angle PU_1C$.

Also $\angle BAU_1 = \angle BCU_1$.

$\therefore \angle BAT - \angle PAU_1 = 180° - \angle PU_1C - \angle BCU_1$.

$\therefore U_1AT = 180° - 90° = 90°$ for $\angle PU_1C$ is measured by $\frac{1}{2}$ arc PC and $\angle BCU_1$ is measured by $\frac{1}{2}$ arc BU_1.

But PP_1C, which is a right $\angle$, is measured by $\frac{1}{2}$ arc $(PC + BU_1)$.

$\therefore U_1A \perp AT$ and so Simson's line of P must be. In like manner we can prove Simson's line of $T \perp AP$.

Now, if Q is the point on the circumference opposite P, then AU_1 and AQ are isogonal conjugate lines, for

$\angle U_1AT = \angle QAP = 90°$ and

$\angle TAC = \angle BAP$ with $\angle U_1AQ$ common.

$\therefore \angle U_1AT - \angle QAU_1 - \angle TAC = \angle QAP - \angle U_1AQ - \angle BAP$.

$\therefore \angle BAU_1 = \angle CAQ$.

4. If P and Q are opposite points on the circumference, their Simson's lines are $\perp$ to each other.

Now, the isogonal conjugate of AP is $\perp$ to isogonal conjugate of AQ, and, therefore, since the Simson's line of $P \parallel AU_1$ and the Simson's line of Q AT, the Simson's line of P will be $\perp$ to Simson's line of Q.

5. A side, BC, and its altitude in a triangle are the Simson's lines of A′ and A, respectively, where A′ is the point on the circumference opposite A. (Fig 4.)

Since the feet of the $\perp$ s from A to AB and AC coincide with A, and the foot of the $\perp$ from A to BC is H_a, therefore the Simson's line of A is AH_a. Again, the feet of $\perp$ s from A′ to the sides AB, AC, and BC are B, C, and A'_a, respectively ; hence BC is the Simson's line of A′.

Since AH_a, BH_b and CH_c are the Simson's lines of A, B, and C, respectively, their Simson's lines concur in H, the ortho-center.

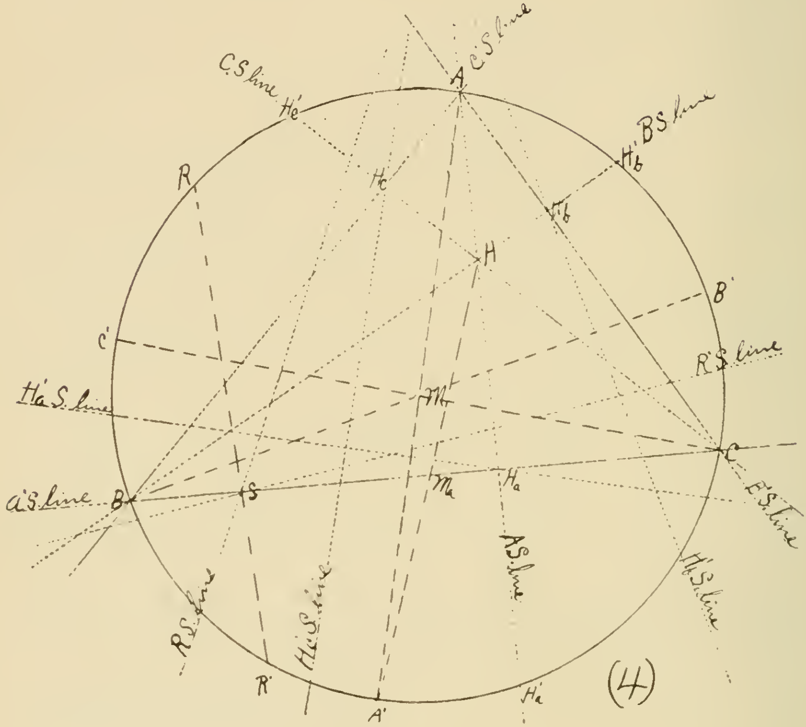
6. Let A′, B′ and C′ be the points on the circumference opposite A, B and C, respectively, and H_a', H_b', and H_c' be the points where AH_a, BH_a and CH_a, produced, cut the circumference, then the

Simson's lines of A, B′ and C′ concur in A,

Simson's lines of B, C′ and A′ concur in B, and

Simson's lines of C, A′ and B′ concur in C,

Also, since the Simson's line of H_a', H_b' and H_c' must pass through H_a, H_b and H_c, respectively, we have the



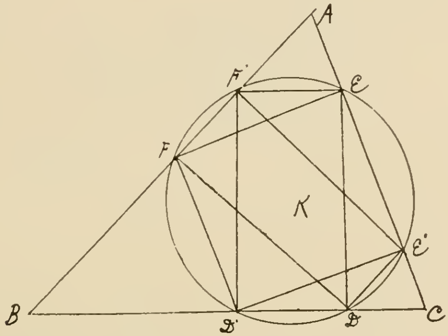
Simson's lines of A, A′ and H_a' concurring in H_a,

Simson's lines of B, B′ and H_b concurring in H_b and

Simson's lines of C, C′ and H_c' concurring in H_c.

Since the point of concurrency of the Simson's lines of the extremities of a chord ⊥ to BC is the point where this chord intersects BC, it follows that the Simson's lines of the extremities of all chords ⊥ to BC are concurrent with Simson's line of A′; the Simson's lines of extremities of all chords ⊥ AC are concurrent with Simson's line of B′; and the Simson's lines of the extremities of all chords $\perp$ AB are concurrent with Simson's line of C′. Thus there is a triple infinity of sets of three points on the circumcircle, the points of concurrency of the Simson's lines of which lie in the sides of the fundamental triangle.

7. Since, in the cosine circle (Fig. 6), F′ EDD′, FEE′ D′, and FF′ E′ D are all rectangular, it follows at once that the Simson's line of D′, with regard to rt. △ DEF, is FD, of E′, DE and of F′, EF. Also Simson's line of D, with regard to rt. △ D′E′F′, is D′ E′, of E, E′ F′ and of F, F′D′.



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8. In Fig. 2, M_a, M_b, M_c are the midpoints of the sides of fundamental triangle opposite A, B, and C, respectively. H_a'', H_b'', H_c'' are the midpoints of AH, BH and CH, respectively, where H is the ortho-center.

Now $M_bA = M_bH_a$.

$\therefore \angle M_bH_aA = \angle M_bAH_a$.

Likewise $\angle M_cH_aA = \angle M_cAH_a$.

$\therefore \angle M_cH_aM_b = \angle A$.

We also know $\angle M_cM_aM_b = \angle A$.

$\therefore$ M_c, M_b, M_a and H_a are concyclic.

In the same way we can show

M_c, M_b, M_a and H_b and M_c, M_b, M_a, and H_c to be concyclic.

$\therefore$ Since three points determine a circle, these six points are all concyclic.

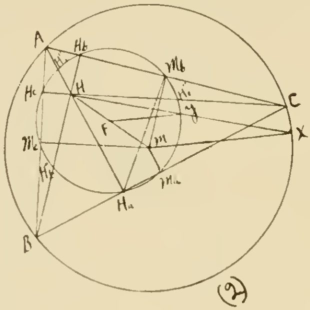
Now H_a, H_b, H_c are the feet of the altitudes of △ AHB.

But we have just shown that, in any triangle, the feet of its altitudes and the midpoints of its sides are all concyclic.

$\therefore$ H_a, H_b and H_c and H_a'', H_b'', and H_c'' are concyclic.

Then, since three points determine a circle, M_a, M_b, M_c, H_a, H_b, H_c, H_a'', H_b'' and H_c'' must all lie on the same circle. This circle, since it passes through nine definite points, is called the *nine-point* circle.

A $\perp$ to the midpoint of $H_a M_a$ meets HM at its midpoint, say F. So $\perp$s to $H_b M_b$ and $H_c M_c$ at their midpoints meet HM in F.



∴ F, the midpoint of HM, is the centre of the nine-point circle.

Since it is the circumcircle of rt. △ $M_aM_bM_c$, which has just half the dimensions of △ ABC, its radius will be just half the radius of the larger circle.

9. The nine-point circle bisects any line drawn from H to the circumcircle of the fundamental triangle.

Let MX and FY be any two radii of the two circles. Now, since F is the mid-point of MH and $FY = \frac{1}{2} MX$, HYX is a straight line with Y as its mid-point.

10. Simson's line of P bisects PH. (Fig. 7.)

Let us suppose that D is the midpoint of PH. Then D lies on the nine-point circle. Then we must prove it lies on $P_1 P_2$.

Since $P P_1$ and AH are $\perp$ to BC, they are $\parallel$. Also since D is midpoint of PH, △ P, DH_a is isosceles. Let E be midpoint of AH. Then $DE = \frac{1}{2} AP$.

D, E, and H_a are on the nine-point circle.

A, P, and C are on the circumcircle.

Then since $DE = \frac{1}{2} AP$ and radius of nine-point circle $= \frac{1}{2} R$, $\angle EH_a D$, inscribed in nine-point circle, $= \angle ACP$, inscribed in circumcircle.

$\angle EH_a D = \angle ACP = \angle DP_1 P$.

Let the intersection of P_1 D and AC be P_2,

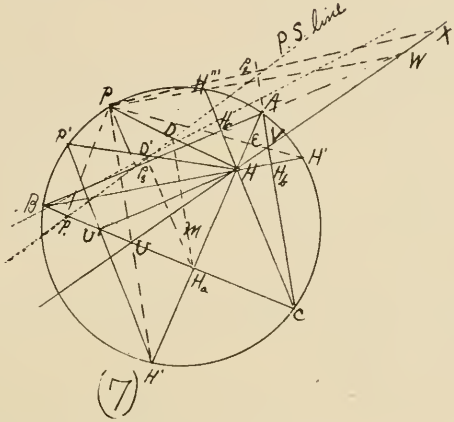
Then $\angle PP_1 D = \angle ACP = \angle PCP_2$,

. ·. P, P_1, C, and P_2 are concyclic and . ·. $\angle PP_2C = \angle PP_1C = 90°$ and $PP_2 \perp AC$.

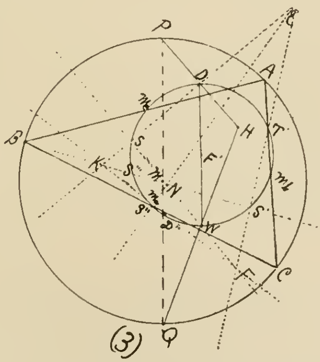
. ·. P_1 D is Simson's line of P.

The point where PH cuts Simson's line of P is called its *center*.

11. The line joining A′, the point opposite the vertex A of △ ABC (Fig. 4), with H, the ortho-centre, bisects the side BC.



For, the Simson line of A′ is BC, . ·. BC bisects A′ H, and, as we have shown, this bisection is on the nine-point circle. But the nine-point circle cuts BC at two places only, at H_a and M_a; hence it is obvious that A′ H passes through M_a, and thus it bisects BC.



12. S, the intersection of the Simson lines of P and Q, the extremities of any diameter of the circumcircle, lies on the nine-point circle. (Fig. 3.)

Take W as the midpoint of QH and D of PH. Then they are both on the nine-point circle and WD must be its diameter, since it is ∥ and equal to $\frac{1}{2}$ QP. Then, as ∠ S is a right ∠, S must also lie on the nine-point circle. S is called the *vertex* of either Simson line.

13. H′, H″ and H‴ (Fig. 7) are the points on the circumcircle through which AH, BH and CH, respectively, pass.

U is the point where PH′ cuts BC.

V is the point where PH″ cuts AC.

W is the point where PH‴ cuts AB.

Now, U, V, H and W lie on a straight line ∥ to the Simson line of P.

$\angle VHH_b = \angle VH''H_b$, ($H_b$ is midpoint of HH″.)

$= \angle PH''B = \angle PCB.$

$\angle UHH_a = UH'H_a = \angle PH'A = \angle PCA.$

Also H, H_a, C, and H_b are concyclic.

$\therefore \angle H_aHH_b + \angle C = 180°.$

But we have just proven $\angle VHH_b + \angle UHH_a = \angle C.$

$\therefore \angle H_aHH_b + \angle VHH_b + UHH_a = 180°$, and

∴ U, H and V are collinear.

Now, $\angle WHH''' = WH'''H = 180° - \angle PH'''C.$

$= \angle B + \angle UH'H.$

$= \angle B + \angle UHH'.$

Also B, H_a, H, and H_c are concyclic.

$\therefore \angle B + \angle H_aHH_c = 180°$ and

$\therefore \angle B + \angle UHH' + \angle H_cHU = 180°.$

So then $\angle WHH''' + \angle H_cHU = 180°$, which proves W, H and U collinear.

Therefore all four points, W, V, H and U must be collinear.

Now, PP_2 is ∥ to HH″, for both are ⊥ to AC.

∴ △ PVX is isosceles, and $PP_2 = P_2X$.

Now, $\angle P_2XV = \angle PCP_1$ and P, P_2, C, and P_1 are concyclic.

$\therefore \angle PP_2P_1 = \angle PCP_1 = \angle P_2XV.$

∴ $P_1P_2P_3$ is ∥ to UVW.

From this we can also see that Simson's line of P bisects all lines from P to the line WVHU.

14. The angle between the Simson lines of two points P and P′ is equal to an ∠ inscribed in the circumcircle with PP′ an arc and also to an angle inscribed in the nine-point circle with arc equal to the part of the circumference included between the centers of their Simson's lines.

Draw $P'H'$ (Fig. 7), letting it cut BC in U'. Then, from above proposition, $HU' \parallel$ to Simson's line of P'. Also $HU \parallel$ to Simson's line of P.

$\therefore \angle (S, S') = \angle (HU, HU')$. Now, H is the center of similitude of the circumcircle and the nine-point circle. Draw $P'H$, letting it cut the Simson line of P' at D'.

Then P and D, P' and D' and H' and H_a are corresponding points.

$\therefore H_aD$ and H_aD' and $H'P$ and $H'P'$ are $\parallel$ lines, whence $\angle DH_aD' = \angle PH'P' = \angle U'HU$.

15. If P and Q (Fig. 1) be the extremities of a diameter and R and R' two other opposite points such that PR and QR' are $\perp$ to Simson's line of P and PR' and QR are $\perp$ to Simson's line of R, then the Simson's line of R is parallel to PQ and Simson's line of R' is $\perp$ to PQ.

Since the angle between the Simson's lines of two points is equal to an angle inscribed in the arc between them, we know that $\angle ZXY = \angle YQZ$. Also $ZX \parallel QY$.

$\therefore$ QZXY is a parallelogram, and XY, the Simson's line of R, is $\parallel$ to PQ. Then it follows that Simson's line of R is $\perp$ to PQ, since it is conjugate to Simson's line of R.

16. If ES and FS (Fig. 3) are the Simson's lines of opposite points on the circumcircle, and EF be any other Simson's line, then $T'E = T'F = T'S$, where T' is the center of the last named Simson's line.

$\angle T'ED = \angle T'SD$ (a previous proposition).

$\therefore T'E = T'S$. In like manner we can show $T'F = T'S$. $\therefore T'E = T'F$.

The Simson's lines of opposite points on the circumcircle are said to be *conjugate*.

17. The arc between the vertices of two Simson's lines (not conjugate) is twice as large as the arc between their centers. For $ET'S$ is an isosceles $\triangle$ and $\angle ET'S = 2\ T'ST$. But $ET'S = S'T'S$. $\therefore$ arc $SS' = 2$ arc $T'T$. Now suppose $T'S$ is less than R (it never can be greater), then S'' could be another point on the nine-point circle such that $T'S'' = T'S$ and $\angle ES''F$ would also be a right $\angle$. It is thus evident that there are always two pairs of conjugate Simson's lines passing through E and F.

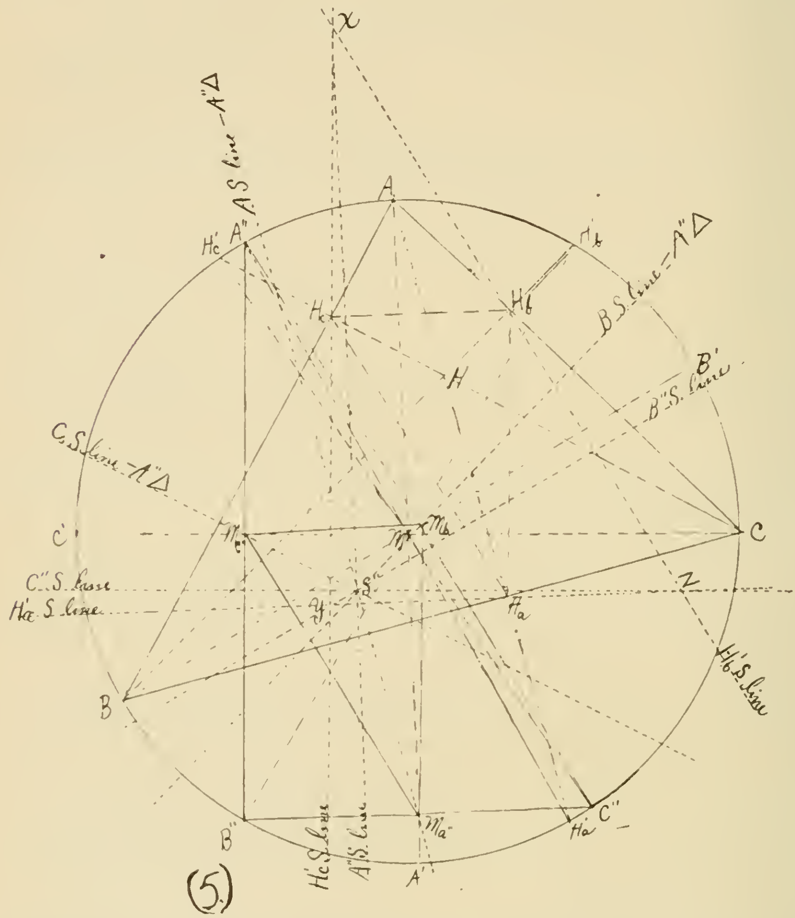
The limit of EF is 2R.

For when S and S'' coincide at S''', $T'S''' = T'E = T'F = r$. In this case we have but one pair of conjugate Simson's lines.

18. If two Simson's lines, SD and $S''D''$, which are not conjugate, cut a third Simson's line, $T'S'$, at equal distances, E and F, from its center, T', then

the line joining the point of intersection, K, of SD and S″D″ with S′, the vertex of T′S′ is a Simson's line conjugate to T′S′.

Let ES and FS″ intersect at K, and ES″ and FS intersect at N (Fig. 3). Since the pair of lines, ES, FS and ES″, FS″ are conjugate Simson's lines, they



are ⊥ to each other, or ES″ and FS are altitudes in the △ EKF. Therefore KN is the third altitude, and we may prove S′ to be the foot of this altitude on EF.

The nine-point circle of the △ ABC passing through the feet, S and S″, of the two altitudes, and through the middle point, T′, of one side of the △ EKF, must be also the nine-point circle of the $\triangle$ EKF, and therefore the second intersection of the nine-point circle with the side, EF, must be the foot of the altitude to EF, or KS′.

Hence S′ is the foot of the altitude KN. But any side and its altitude is a pair of conjugate Simson's lines, and since EF is a Simson's line of a point on the circumcircle of ABC, KN is the Simson's line conjugate to EF.

Any triangle like EKF formed by three Simson's lines, the altitudes of which are Simson's lines conjugate to the sides, and having the nine-point circle in common with the triangle ABC, we shall call a *Simson Triangle.*

Since the nine-point circle is common to both triangles ABC and EKF, the radius of the nine-point circle is one-half the radius of the circumcircle of either triangle; therefore the radius of the circumcircle of any Simson triangle is equal to the radius of the circumcircle of the original triangle.

19. The common vertex S‴ of the pair of limiting Simson's lines belonging to TS is on the same straight line as K, N and S′. For, since T′S‴ is a diameter of the nine-point circle, $\angle$ T′S′S‴ $= 90°$, or S′S‴ $\perp$ to EF. $\therefore$ S‴ is on the altitude KS′.

20. A″, B″ and C″ are points on the circumference opposite H_a', H_b' and H_c' respectively. (Fig. 5.) Prove that the Simson's lines of A″, B″, and C″ are | respectively to AA′, BB″ and CC″ and that they are $\perp$ respectively to B″C″, A″C″, A″B″. Now the angle between the Simson's lines of A and A″ will be equal to an angle measured by $\frac{1}{2}$ arc AA″. But the angle between AA′ and AH_a, the Simson's line of A, is measured by an arc equal to this. Therefore Simson's line of A″ is || to AA′. So the Simson's line of B″ is || to BB′ and Simson's line of C″ || to CC′.

Now arc H_b' CB″ = arc H_c' BC″ = 180°.

$\therefore$ arc H_b' CC″ = arc H_c' BB″

$\therefore$ H_b' H_c' || B″ C″, also H_b' H_a' || A″ B″ and

H_a' H_c' || A″C″. Therefore $\triangle$ A″ B″ C″ is equivalent to $\triangle$ H_a' H_b' H_c', being inscribed in same circle and having sides equal and parallel.

$\angle$ H_c'CA = $\angle$ H_b' BA. (From similarity of $\triangle$s ABH_b and ACH_c).

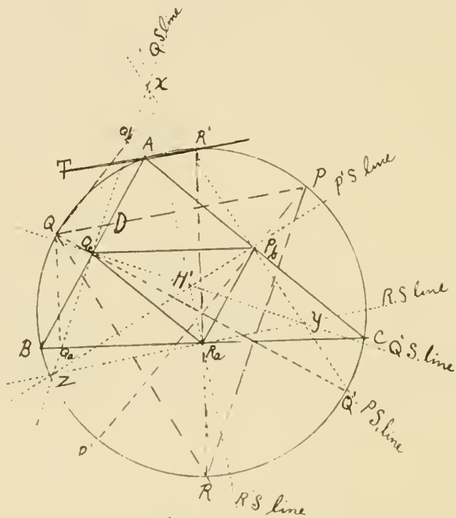
$\therefore$ arc AH_b' = arc AH_c', $\therefore$ since AA′ is a diameter it must be $\perp$ to chord H_c' H_b' and therefore to B″ C″. So we may prove BB′ $\perp$ A″ C″ and CC′ $\perp$ A″ B″.

21. The Simson's lines of H_a' H_b' H_c' form a Simson's triangle XYZ of which the Simson's lines of A″, B″, C″ are the altitudes, A″, C″, B″ being points opposite H_a', H_b', H_c' respectively.

Let Simson's lines of H_b', H_c', and H_c', H_a' and H_a', H_b' concur in X, Y, and Z respectively.

The Simson's lines of A'' and H_a', of B'' and H_b', and of C'' and H_c' are conjugate, therefore their intersections, u, v, and w will lie on the nine-point circle of rt △ ABC. The Simson's lines of H_a', H_b' and H_c' must pass through H_a, H_b, H_c respectively and therefore rt △ XYZ must have the same nine-point circle as △ ABC. Now since H_a, H_b and H_c can not be the feet of altitudes they must be the midpoints of the sides and therefore u, v and w must be the feet of altitudes.

Thus the four points of concurrency are established, namely X, Y, Z and S''. S'', being formed by the intersection of the altitudes of △ XYZ, is the orthocenter of the same.



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22. If R, P and Q (Fig. 8) be taken as the midpoints of arcs BC, AC and AB, respectively, and R', P' and Q' be the points on the circumference opposite R, P, and Q, then the Simson's lines of R, P, and Q will form a △ XYZ, the altitudes of which will be the Simson's lines of R', Q' and P'.

It may be assumed that XYZ is the triangle formed by the intersection of the Simson lines of R, P and Q. That the Simson's line of Q' is the altitude on side XZ may be established thus:

$\angle AQ_bQ_c = \angle A - \angle AQ_cQ_b$.

But, since Q, Q_c, A, and Q_b are concyclic, $\angle AQ_bQ_c = \angle AQQ_c$, which is measured by $\frac{1}{2}$ arc AQ'.

Also, since Q, B, Q_a, and Q_c are concyclic, $\angle BQQ_c = 180° - \angle BQ_aQ_c$.

$\therefore \angle Q_cQ_aC = \angle BQQ_c$, which is measured by $\frac{1}{2}$ arc BQ'. But arc $BQ' =$ arc AQ'.

(A.) $\therefore \angle AQ_bQ_c = \angle CQ_aQ_c$.

But $\angle AQ_bQ_c = \angle A - \angle AQ_cQ_b$, and

$\angle CQ_aQ_c = \angle B + \angle BQ_cQ_a$.

$= \angle B + \angle AQ_cQ_b$.

Now the Simson's line of Q' is $\perp$ to Q_bQ_a the Simson's line of Q, at Q_c, and $\angle AQ_cP_b = \angle B$ and $\angle BQ_cR_a = \angle A$. And from (A.) we know that $\angle Q_bQ_cP_b = \angle Q_aQ_cR_a$, $\therefore \angle P_bQ_cH', = \angle R_aQ_cH'$ and thus Simson's line of Q' is proven to be the internal bisector of $\angle Q_c$ in rt. $\triangle R_aP_bQ_c$. In like manner the Simson's line of R' may be proven to be the internal bisector of $\angle R_a$, and Simson's line of P', the internal bisector of $\angle P_a$ of same triangle. Since the Simson's lines of Q, R, and P are $\perp$ to these internal bisectors at the vertices of the $\triangle$, they must be the external bisectors of the same angles. But we know that the internal and external bisectors of the $\angle$s of a triangle concur in four points. Thus X, Y, Z and H' are each points of concurrency of three Simson's lines. H', of course, is the ortho-center of $\triangle$ XYZ.

23. $\triangle$ XYZ has the same nine-point circle as $\triangle$ ABC and is therefore inscribable in the same sized circle as rt. $\triangle$ ABC.

That it has the same nine-point circle follows easily from the fact that three points that must lie on its nine-point circle, the feet of its altitudes, are coincident with three points on the ABC nine-point circle, the midpoint of its sides. Since three points determine a circle, their nine-point circles must be identical.

24. The Simson's line of P is $\parallel$ to RQ, Simson's line of R is $\parallel$ to PQ, and Simson's line of Q is $\parallel$ to RP.

The Simson's line of R will be $\perp$ to Simson's line of R', so let us prove Simson's line of R' $\perp$ to PQ'.

Now, the Simson's line of R' will be $\perp$ to the line isogonal conjugate to AR' which we call AT. Then let us prove AT $\parallel$ to PQ.

To be $\parallel$ $\angle ADP$ must $= \angle CAR'$.

$\angle ADP$ is measured by $\frac{1}{2}$ arc $(AP + BQ)$.

$\angle CAR'$ is measured by $\frac{1}{2}$ arc CR'.

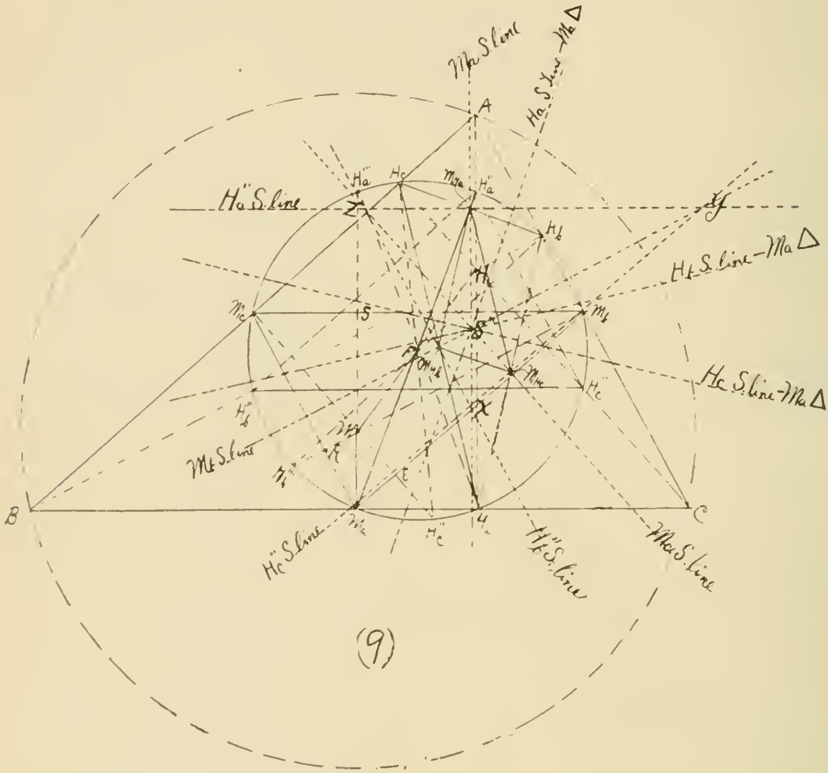
Arc $AP =$ arc CP, and $\frac{1}{2}$ arc BQ measures $\frac{1}{2} \angle C$.

Now, $\angle$ PRR$' = 90° -$ PR$'$R.

$= 90° - \frac{1}{2}(A + B)$.

$= \frac{1}{2}$ C.

$\therefore$ arc BQ = arc PR$'$, and so $\angle$ ADP = $\angle$ CAR$'$. Hence AT is PQ and $\therefore$ Simson's line of R is to PQ. In like manner we may prove Simson's line of P to RQ, and Simson's line of Q to RP.



(9)

25. The Simson's lines of H_a'', H_b'', H_c'' (Fig. 9) with respect to $\triangle$ $H_aH_bH_c$, inscribed in the nine-point circle, form a rt. $\triangle$ XYZ, the altitudes of which are the Simson's lines of M_a, M_b and M_c with respect to the same rt. $\triangle$.

H, the orthocenter of $\triangle$ ABC, is the incenter of $\triangle$ $H_aH_bH_c$, for $\angle$s AH_cH and AH_bH being right $\angle$s, A, H_c, H and H_b are concyclic with AH as diameter.

H_a'', being midpoint of AH, is the center and therefore chord $H_a''H_b$ = chord $H_a''H_c$. Therefore H_aH_a'' bisects $\angle H_a$, H_bH_b'' bisects $\angle H_b$, and H_cH_c'' bisects $\angle H_c$.

H_a'', H_b'' and H_c'' are points on the nine-point circle opposite M_a, M_b, M_c, respectively, for, since MM_a and H_aH_a'' are and the center of the nine-point circle is F, the midpoint of MH, a line from M_a through F, will meet AH_a on the circumference of the nine-point circle, necessarily at H_a''. In like manner H_b'' can be shown to be opposite M_b, and H'' opposite M_c.

This proves $H_a''H_b''$ | to M_aM_b and equal to it, and therefore | to AB, $H_b'H_c'$ equal to M_bM_c and parallel to both M_bM_c and BC, and $H_c''H_a''$ equal to M_cM_a and parallel to both M_cM_a and CA.

Now, the Simson's line of M_a will be $\perp$ to BC and | to AH_a, for H_aC is isogonal conjugate to H_aM_a since AH_a bisects $\angle H_bH_aH_c$, thus making $\angle CH_aH_b = \angle M_aH_aH_c$. Therefore the Simson's line of H_a'', since it is conjugate to Simson's line of M_a, is | to M_cM_b, $H_b''H_c$ and BC. In like manner we can prove the Simson's line of $M_b \perp$ AC and | to BH_b and, therefore, the Simson's line of H_b'' | to M_aM_c, $H_c''H_a''$ and AC; and Simson's line of $M_c \perp$ AB and | to CH_c and, therefore, the Simson's line of H_c'' | to M_aM_b, $H_b''H_a''$ and AB.

Now $\triangle M_{Ha}M_{Hb}M_{Hc}$ is oppositely similar to $\triangle H_aH_bH_c$. Also, from what we have proven before, the Simson's lines of M_a and H_a'' must both pass through M_{Ha}. Then the Simson's line of M_a being || to AH_a, will bisect $\angle M_{Hb}M_{Ha}M_{Hc}$. In like manner we can show that the Simson's lines of M_b and M_c bisect $\angle$s $M_{Ha}M_{Hb}M_{Hc}$ and $M_{Ha}M_{Hc}M_{Hb}$. Now the Simson's lines of H_a'', H_b'' and H_c'' are $\perp$ to Simson's lines of M_a, M_b and M_c, respectively, and therefore they are the external bisectors of the $\angle$s of the rt. $\triangle M_{Ha}M_{Hb}M_{Hc}$. Since the internal and external bisectors of a $\triangle$ meet by threes in four points, we may conclude the Simson's lines of M_a, H_b'', and H_c'' concur in X, Simson's lines of M_b, H_c'', and H_a'' concur in Y, Simson's lines of M_c, H_a'', and H_b'' concur in Z, and Simson's lines of M_a M_b M_c concur in S'''. S''', we see, is the orthocenter of $\triangle$ XYZ and in-center of $\triangle M_{Ha}M_{Hb}M_{Hc}$.

$\triangle$ XYZ has its sides || to sides of $\triangle M_aM_bM_c$ and oppositely || to $\triangle$s $H_a''H_b''H_c''$ and ABC.

26. Let us prove $\triangle$ XYZ equivalent to $\triangle$s $M_aM_bM_c$ and $H_a''H_b''H_c''$ and, therefore, inscribable in the nine-point circle.

$\triangle M_{Ha}M_{Hb}M_{Hc}$ bears the same relation to $\triangle$ XYZ that $\triangle H_aH_bH_c$ does to $\triangle$ ABC. Therefore, since $\triangle M_{Ha}M_{Hb}M_{Hc}$ is $\frac{1}{4}$ the size of $\triangle H_aH_bH_c$, $\triangle$ XYZ must be $\frac{1}{4}$ the size of rt. $\triangle$ ABC and thus equivalent to $\triangle M_aM_bM_c$, and rt. $\triangle H_a''H_b''H_c''$ and hence inscribable in the nine-point circle of the fundamental $\triangle$.

27. If $H_a'''\,H_b'''\,H_c'''$ are the points on the nine-point circle opposite $H_a\,H_b$ and H_c respectively, then the Simson's lines of M_c, H_c'', and H_c''' concur in M_{Hc}, for $H_a\,H_b$ is Simson's line of H_c'''. Likewise the Simson's lines of M_b, H_b'' and H_b''' concur in M_{Hb} and the Simson's lines of M_a, H_a'', and H_a''' concur in M_{Ha}.

28. Now considering $M_a\,M_b\,M_c$ as the reference $\triangle$ in the nine-point circle, let us prove that the Simson's lines of these same points, *i. e.*, M_c, H_c'' and H_c''', etc., concur.

Chord $M_c\,H_c'''$ is $\parallel$ $H_c\,H_c''$ and $\therefore$ $\perp$ to $M_a\,M_b$. This is true because $\angle$ $H_c\,M_c\,H_c'''$ is a right $\angle$.

The Simson's line of M_c will be this chord $M_c\,H_c'''$, the Simson's line of H_c''' will pass through E, the foot of this altitude, and the Simson's line of H_c'' will be side $M_a\,M_b$ since it is a point opposite the vertex M_c.

So, also, the Simson's lines of M_b, H_b'' and H_b''' will concur in R, and the Simson's lines of M_a, H_a'', and H_a''' concur in S.

29. Now by noticing the lettering and arrangement of (Fig. 17) it will be seen that H_a''', H_b''' and H_c''' correspond to H_a', H_b' and H_c' of that figure, and that H_a, H_b, and H_c correspond to A'', B'', and C'' of that figure. Therefore we know at once that the Simson's lines of H_a, H_b and H_c and of H_a''', H_b''' and H_c''' concur just as in that case by threes in four different points, the point of concurrency of H_aH_b and H_c being the ortho-center S''' of the triangle formed by the intersection of the Simson's lines of the other three points.

30. Also since $H_a''H_b''H_c''$ and points H_a, H_b and H_c bear the same relation to the nine point circle that $\triangle$ ABC as points H_a', H_b', and H_c' do to the circle in Fig. 5, it follows at once that what was true of the Simson's lines of those points is also true of these.

Depending upon this same comparison between Figs. 5 and 9 it follows that the Simson's lines of points A, B and C in Fig. 5 concur at in-center of rt. $\triangle$ $M_{a''}M_{b''}M_{c''}$.

31. Now let us prove that S'' (Fig. 5), the point of concurrency of Simson's lines of A'', B'', C'', is also the in-center of $\triangle$ $M_{a'}M_{b''}M_{c''}$.

We have already proven that the Simson's lines of H_a', H_b' and H_c' form a Simson triangle XYZ, of which the Simson's lines of A'', B'' and C'' are the altitudes, and that $\triangle$ XYZ, is equivalent to $\triangle$ $A''B''C''$ with their sides respectively $\parallel$. Now, since H_a, H_b and H_c are the midpoints of the sides of $\triangle$ XYZ, $\triangle$ $H_aH_bH_c$ will be equal in every respect and similarly placed to $\triangle$ $M_{a''}M_{b''}M_{c''}$. We have also proven H to be the in-center of this $\triangle$ $H_aH_bH_c$. Again, since the Simson's lines of A'', B'' and C'' bisect A'' H, B'' H and C'' H, respectively, it is clear that if $\triangle$ XYZ were to be given the rank of rt. $\angle$ $A''B''C''$ and a new one were to be formed from it, as it is formed from $\angle$ $A''B''C''$, then the point S'' would fall upon H. Therefore, since H is the in-center of $\triangle$ $H_aH_bH_c$, S'' must be the in-center of $\triangle$ $M_{a''}M_{b''}M_{c''}$.

Therefore we see that the six Simson's lines, three with reference to one $\triangle$ and three with reference to the other, meet in the same point.

32. This, at the same time, establishes another even more interesting proposition, namely: If the Simson's lines of the vertices of a first $\triangle$ with reference to a second $\triangle$ concur in a point S'', then the Simson's lines of the vertices of the second $\triangle$ with reference to the first $\triangle$ concur in the same point S''.

The broad scope covered by this proposition would enable me to double in number the points of concurrency of Simson's lines, but there would be little benefit in merely pointing them out, as the interested reader can easily see them for himself.

A Bibliography of Foundations of Geometry. By Morton Clark Bradley.

Euclid's treatment of parallels and angles and his definitions and axioms—particularly his twelfth—are the points of controversy that cause the most discussion. For nearly twenty centuries Euclid's work remained unquestioned. Since John Kepler's day, however, there have been new theories constantly advanced, theories built on axioms and definitions, a part of which, at least, are different from those of Euclid. The most important of the non-Euclideans are John Bolyai, Lobatschevski, Helmholtz, Riemann, Clifford, Henrici, Caley, Sylvester and Ball. The most prominent exponent of the non-Euclidean ideas in this country is Prof. Geo. Bruce Halsted, of Texas University. These mathematicians hold that Euclid's twelfth axiom is not, strictly speaking, an axiom—that it is not "a self-evident and necessary truth," but that it requires demonstration. They claim, too, that his definitions are not sufficient nor necessarily intelligible. Some of these men have built up new theories upon their substituted axioms and definitions, retaining those of Euclid that fit their theories. A few of these "reform" works are mere quibbles on words, but others deserve the serious consideration of all interested in pure geometry.

The list following is a complete list of English references to be found in the mathematical library of the University of Indiana or in the private