Take a circle whose diameter is thirty-six units; Angel's method makes the side of a 36-gon equal to 3.33982 units, while the true length is 3.13776 units. The larger number of sides makes the error of the method more apparent.

Concurrent Sets of Three Lines Connected with the Triangle.

By Robert Judson Aley.

To the student of the pure geometry of the triangle, few subjects are more interesting than the concurrency of lines. The following collection of concurrent sets of three lines has been made in the hope that it may prove of value to geometric students. No claim is made to completeness. The list is as complete as the author could make it with the material to which he had access. Many of the notes, and a large number of the propositions have been taken from the published papers of Dr. J. S. Mackay, of Edinburgh, perhaps the foremost student of the geometry of the triangle. No classification of the propositions seems possible and so none has been attempted.

1. The median lines of a triangle are concurrent. The point of concurrency, usually denoted by G, is called the *median point* or centroid.

2. The in-symmedian lines of a triangle are concurrent. The point of concurrency is called the symmedian point or Grebe's point, and is generally denoted by K. (For a history of this point, see J. S. Mackay, in Proceedings of Edinburgh Mathematical Society, Vol. XI.)

3. The altitudes of a triangle are concurrent. The point of concurrency, usually denoted by H, is called the ortho centre. (This proposition occurs in Archimedes's Lemmas and in Pappus's Mathematical Collection.)

4. The internal angle bisectors of a triangle are concurrent. The point of concurrency is the center of the inscribed circle and is usually denoted by I. (Euclid IV, 4.)

5. The internal bisector of any angle of a triangle and the external bisectors of the other two angles of the triangle are concurrent. The points of concurrency, denoted by I_1, I_2, I_3 are the centers of the three escribed circles.

6. The perpendiculars to the sides of a triangle at the midpoints of the sides concur at the center of the circumscribed circle. This point of concurrence is usually denoted by O. (Euclid.)

7. Lines drawn from the vertices to the points of contact of the in-circle with the opposite sides are concurrent. (The point of concurrency, Γ, is called the Gergonne Point. It was named by J. Neuberg after J. D. Gergonne.)

8. Lines drawn from the vertices to the points of contact of the escribed circles with the opposite sides concur at Q, Nagel's Point. (For a number of interesting properties of this point, see Schwatt's "Geometric Treatment of Curves.")

9. Lines drawn from the vertices making equal angles with the sides AB, BC, CA, respectively, concur at Ω and Ω', the two Brocard points of the triangle.

10. If A_1, B_1, C_1, is Brocard's first triangle, then AA_1, BB_1, CC_1, concur at D, the point isotomic conjugate to K.

11. If L, M, N, be the midpoints of the sides of the triangle ABC, and L', M', N' the midpoints of the sides of the triangle $A_1B_1C_1$, then LL', MM' and NN' concur at S.

12. AL', BM', and CN' concur at S'. S and S' are isogonal conjugate points. (Schwatt's "Geometric Treatment of Curves," p. 5.)

13. Perpendiculars from A, B, C upon B_1C_1, C_1A_1, A_1B_1, respectively, concur at N, a point on the circumcircle of the triangle ABC, known as *Tarry's Point.*

14. Lines through A, B, C, parallel to B_1C_1, C_1A_1, A_1B_1, respectively, concur at a point on the circumcircle of the triangle ABC known as *Steiner's Point.*

15. Parallels to AB and CA through C and B, respectively, concur with the median through A. There are evidently three such points of concurrency. These points are sometimes called the *external median points.*

16. If three lines through the vertices are concurrent, their isogonal conjugates with respect to the angles of the triangle are also concurrent. (Steiner's Gesammelte Werke I., 193, 1881.) If the ratios of the distances of the first point from the sides are $l:m:n$, those of the second point are

$$\frac{1}{l} : \frac{1}{m} : \frac{1}{n}.$$

17. Perpendiculars to the sides of the triangle ABC from the midpoints of the sides of the orthic triangle of ABC are concurrent. (Edouard Lucas in Nouvelle Correspondance Mathématique II, 95, 218, 1876.)

18. The ex-symmedians from any two vertices and the in-symmedian from the third vertex are concurrent. There are evidently three such points of concurrency. They are sometimes called the *external symmedian points.*

19. If three lines drawn from the vertices of a triangle to intersect the opposite sides are concurrent, the lines isotomic conjugate to them are also concurrent. If the ratios of the distances of the first point of concurrency from the sides are $l:m:n$, the ratios of the second point are

$$\frac{1}{a^2l} : \frac{1}{b^2m} : \frac{1}{c^2n}.$$

20. If the three perpendiculars from the vertices of one triangle upon the sides of another triangle are concurrent, then the three perpendiculars from the vertices of the latter upon the sides of the former are also concurrent. (Steiner, Gesammelte Werke I., 157, 1881.) (Lemoine calls such triangles *orthologous* and the points of concurrency *centers of orthology.*

21. Brocard's Triangle and ABC are orthologous. Perpendiculars from A_1, B_1, C_1, upon BC, CA, AB, respectively, are concurrent. (See No. 13.)

22. If three points be taken on the sides of a triangle such that the sums of the squares of the alternate segments taken cyclically are equal, the perpendiculars to the sides of the triangle at these points are concurrent. (T. G. de Oppel, "Analysis Triangulorum," p. 32, 1746.)

23. If on the sides of a triangle ABC, equilateral triangles LBC, MCA, NAB be described externally, AL, BM, CN are equal and concurrent.

24. If on the sides of a triangle ABC, equilateral triangles $L'BC$, $M'CA$, $N'AB$ be described internally, AL', BM' CN' are equal and concurrent. (Dr. J. S. Mackay gives 24 in Vol. XV of Proceedings of Edinburgh Mathematical Society and attributes 23 to T. S. Davies in Gentleman's Diary for 1830, p. 36.)

25. If $A''B''C''$ be Nagel's triangle, then perpendiculars from A, B, and C upon $B''C''$, $C''A''$, $A''B''$, respectively, are concurrent.

26. Perpendiculars from A'', B'', C'' upon BC, CA, AB, respectively, are concurrent.

27. If A', B', C' be the midpoints of the arcs subtended by BC, CA, AB, respectively, then perpendiculars from A', B', C' upon $B''C''$, $C''A''$, $A''B''$, respectively, are concurrent.

28. Perpendiculars from A'', B'', C'' upon $B'C'$, $C'A'$, $A'B'$, respectively, are concurrent.

29. If distances equal to $2r$ (diameter of the inscribed circle) be laid off from the vertices on each of the altitudes, three points A^{iv}, B^{iv}, C^{iv} are obtained. Perpendiculars from A, B, C upon $B^{iv}C^{iv}$, $C^{iv}A^{iv}$, $A^{iv}B^{iv}$, respectively, are concurrent.

30. Perpendiculars from A^{iv}, B^{iv}, C^{iv} upon BC, CA, AB, respectively, are concurrent. (Nos. 25, 27 and 29 are given in Schwatt's "Geometric Treatment of Curves," pages 40, 43 and 44. Nos. 26, 28 and 30 are direct consequences of the orthologous relation of the triangles. See No. 20.)

31. The perpendiculars from the middle points of the sides of Brocard's first triangle upon the corresponding sides of the triangle ABC are concurrent.

32. The lines joining the middle points of the sides of a triangle with those of the segments towards the angles of the corresponding altitudes meet in a point and bisect each other.

33. The straight lines which join the midpoint of each side of a triangle to the midpoint of the corresponding altitude concur at the *symmedian point.* (Dr. F. Wetzig in Schlömlich's Zeitschrift, XII, 289.)

34. If two sides of a triangle are divided proportionally the straight lines drawn from the points of section to the opposite vertices, will intersect on the median from the third vertex.

35. Every two perpendiculars to the sides of a triangle at points of contact of escribed circles external to the same vertex are concurrent with the perpendicular to the opposite side at the point of contact of the inscribed circle. There will be three such points of concurrency.

36. If the three sides of a triangle be reflected with respect to any line, the three lines through the vertices parallel to the reflexions of the opposite sides are concurrent.

37. The vertices of ABC are joined to a point O, and a triangle $A'B'C'$ is constructed having its sides parallel to AO, BO, CO respectively. Lines through A', B', C' parallel to the corresponding sides of the triangle ABC are concurrent.

38. If XYZ be any transversal of the triangle ABC, and if AX, BY, CZ form the triangle PQR, then AP, BQ, and CR are concurrent.

39. If D, E, F be the feet of the altitudes, then the lines connecting A, B, C to the middle points of EF, FD, DE, respectively, concur at the symmedian point.

40. The perpendiculars from A, B, C upon EF, FD, DE are concurrent.

41. Through the vertices of the triangle ABC lines parallel to the opposite sides are drawn, meeting the circumcircle in A', B', C'. $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in P, Q, R, respectively. AP, BQ, CR are concurrent.

42. With the same notation as 41, $A'P$, $B'Q$, $C'R$ are concurrent. (41 and 42 occur in St. John's College Questions, 1890.)

43. Three circles are drawn each touching two sides of the triangle ABC and the circumcircle internally. The points of contact with the circumcircle are L, M, N, respectively. AL, BM, CN are concurrent.

44. If in 43 the circles touch the circumcircle externally in L', M', N', then $L'A$, $M'B$, $N'C$ are concurrent. (43 and 44 are given by Professor de Longchamps, Ed. Times, July, 1890.)

45. If a circle touch the sides of the triangle ABC in X, Y, Z, then the lines joining the middle points of BC, CA, AB to the middle points of AX, BY, CZ, respectively, are concurrent.

46. If a circle cut the sides of the triangle ABC in X, X'; Y, Y'; Z, Z'; if AX, BY, CZ are concurrent, so also are AX', BY', CZ'.

47. If X, Y, Z be three points on the sides of the triangle ABC such that the pencil $D(AC, EF)$ is harmonic, then AD, BE, CF are concurrent.

48. If tangents to the circumcircle at the vertices of the triangle ABC, meet in L, M, N, then AL, BM and CN are concurrent.

49. If on the sides of the triangle ABC, similar isosceles triangles LBC, MCA, NAB be described, AL, BM, CN are concurrent.

50. If the ex-circles touch the sides to which they correspond in D_1, E_2, F_3. the perpendiculars to the sides through these points are concurrent.

51. If D, E, F are the points of contact of the incircle with the sides of the triangle ABC and if DI, EI, FI meet EF, FD, DE in L, M, N, respectively, then AL, BM, CN concur.

52. If DD', EE', FF' are diameters of the incircle through D, E, F, the points of contact with the sides of the triangle ABC, then AD', BE', CF' concur.

53. If P, Q, R be collinear points in the sides BC, CA, AB of the triangle ABC, and if P', Q' R' be their harmonic conjugates with respect to those sides then AP', BQ', CR' are concurrent.

54. If squares $APQB$, $BUVC$, $CXYA$ be described upon the sides of the triangle ABC (all externally or all internally) and if QP meet XY in α, PQ meet VU in β, UV meet YX in γ, then αA, βB, γC concur in K the symmedian point. (Halsted, "El. Synthetic Geometry," p. 150.)

55. $A'B'C'$ is the pedal triangle of Ω, and $A''B''C''$ is the pedal triangle of Ω'. $B''C'$, $C''A'$, $A''B'$ form the triangle XYZ, whose sides are parallel to the sides of ABC. PQR is *the* pedal triangle of ABC. PX, QY, RZ concur at the circumcenter of XYZ.

56. The Simson lines of the median triangle LMN of the triangle ABC, with respect to the vertices P, Q, R of the pedal triangle, concur at the center of Taylor's circle.

57. The Simson lines of the pedal triangle PQR of the triangle ABC, with respect to the vertices L, M, N of the median triangle concur at the center of Taylor's circle.

58. If BW, CV be perpendicular to BC; CU, AW perpendicular to CA; AV, BU perpendicular to AB; then AU, BV, CW concur at the circumcenter of ABC. (C. F. A. Jacobi, "De Triangulorum Rectilineorum Proprietatibus," p. 56.)

59. If triangles $A_1B_1C_1$ and $A_2B_2C_2$ are circumscribed about the triangle ABC in such a manner that their sides are perpendicular to those of ABC, then A_1A_2, B_1B_2, C_1C_2 concur at the circumcenter of ABC. (Probably known

by Jacobi, but not explicitly stated by him. Lemoine stated it in 1873 to the Association Française pour l'Avancement des Sciences.)

60. When three lines through the vertices of a triangle are concurrent, the six bisectors of the three angles they determine intersect with the corresponding sides of the triangle at six points, every three of which on different sides connect concurrently with the opposite vertices if an odd number of them is internal.

61. When three points on the sides of a triangle are collinear, the six bisections of the three segments they determine connect with the corresponding vertices of the triangle by six lines, every three of which through different vertices are concurrent if an odd number of them is internal.

62. When three points on the sides of a triangle are collinear, their three lines of connection with the opposite vertices determine an exscribed triangle whose vertices connect concurrently with those of the original to which they correspond.

63. H_1, H_2, H_3 are points of intersection of AI, BI, CI, respectively, with the inscribed circle. The perpendiculars from H_1, H_2, H_3 upon BC, CA, AB, respectively, are concurrent.

64. The twelve radii from the incenter and the excenters of a triangle, perpendicular to the sides of the triangle, meet by threes in four points which are the circumcenters of the triangles $II_2 I_3$, $II_3 I_2$, $I_3 I I_1$, $I_2 I_1 I$. (I, I_1, I_2, I_3 are the incenter and excenters. See note, p. 99, Vol. I, Proceedings Edinburgh Math. Soc., Dr. Mackay).

65. D, E, F are points of contact of I-circle with the sides of the triangle ABC, D_1, E_1, F_1 points of contact of I_1-circle with sides, D_2, E_2, F_2, of I_2-circle, D_3, E_3, F_3 of I_3-circle.

AD_1, BE_1, CF_1 concur at Γ_1

AD_2, BE_2, CF_2 concur at Γ_2

AD_3, BE_3, CF_3 concur at Γ_3

The points Γ_1, Γ_2, Γ_3 are called the associated Gergonne points. See No. 7.

66. AD_1, BE_3, CF_2 concur at Q_1

AD_3, BE_1, CF_1 concur at Q_2

AD_2, BE_1, CF concur at Q_3

Q_1, Q_2, Q_3, together with Q given in No. 8, are called the Nagel points.

67. AQ, BQ_3, CQ_2 concur at Γ_1.

AQ_3, BQ, CQ_1 concur at Γ_2.

AQ_2, BQ_1, CQ concur at Γ_3.

68. $A\Gamma$, $B\Gamma_3$, $C\Gamma_2$ concur at Q_1.
$A\Gamma_3$, $B\Gamma$, $C\Gamma_1$ concur at Q_2.
$A\Gamma_2$, $B\Gamma_1$, $C\Gamma$ concur at Q_3.

69. AB, DE, D_2E_1 concur at x.
BC, EF, E_3F_2 concur at y.
CA, FD, F_1D_3 concur at z.
x, y, z lie on a line n, say.

70. AB, D_1E_2, D_3E_3 concur at x_1.
BC, E_2F_3, E_1F_1 concur at y_1.
CA, F_3D_1, F_2D_2 concur at z_1.
x_1, y_1, z_1 lie on a line p.

71. AB, NP, I_1I_2 concur at x_2.
BC, PQ, I_2I_3 concur at y_2.
CA, QN, I_3I_1 concur at z_2.
(N, P, Q are the feet of the interior angle bisectors.)
x_2, y_2, z_2 lie on a line q.

72. The three lines n, p, q are concurrent.

73. A', B', C' are the midpoints of the sides of the triangle ABC. Lines drawn through A', B', C', respectively, parallel to the triads of angular transversals which determine Γ, Γ_1, Γ_2, Γ_3, concur at Γ', Γ_1', Γ_2' Γ_3'. Then $\Gamma\Gamma'$, $\Gamma_1\Gamma_1'$, $\Gamma_2\Gamma_2'$, $\Gamma_3\Gamma_3'$ are concurrent at the centroid of the triangle ABC.

74. $I\Gamma'$, $I_1\Gamma_1'$, $I_2\Gamma_2'$, $I_3\Gamma'_3$ concur at the symmedian point of the triangle ABC.

75. IQ, I_1Q_1, I_2Q_2, I_3Q_3 concur at the centroid of the triangle ABC.

(The propositions 65 to 75 inclusive are taken from Mackay's "Euclid" and his "Symmedians and Concomitant Circles.")

76. If DEF be the triangle formed by joining the inscribed points of contact of the triangle ABC; $D_1E_1F_1$ the triangle formed by joining the inscribed points of contact of the triangle DEF; $D_2E_2F_2$ the triangle formed by joining the inscribed points of contact of the triangle $D_1E_1F_1$; I, I_1, I_2, I_3 are the inscribed and escribed centres. I_1D, I_2E, I_3F concur at the homothetic centres of the triangles DEF and $I_1I_2I_3$. ID_1, I_3E_1, I_2F_1 concur at the homothetic centre of the triangles $D_1E_1F_1$ and II_3I_2, and so on. (Dr. Mackay, Proceedings Edinburgh Math. Soc., Vol. I, pp. 51–2.)

77. If three straight lines drawn from the vertices of a triangle are concurrent, the three lines drawn parallel to them from the midpoints of the opposite sides are also concurrent; and the straight line joining the two points of concurrency passes through the centroid of the triangle and is there trisected. (Frigier in Gergonne's Annales, Vol. VII, 170.)

78. If ABC be any triangle and O any point whatever, and A_1, B_1, C_1 be points symmetrical to O with respect to the midpoints of BC, CA, AB, then AA_1, BB_1, CC_1 concur at a point P. The centroid G lies on the line OP and divides it in a constant ratio. (M. d'Ocagne in Nouvelles Annales, Third Series I, 239.)

79. If through K (Grebe's Point) parallels to the sides BC, CA, AB of the triangle ABC are drawn, meeting these sides in D, D'; E, E'; F, F', respectively, and if EF and $E'F'$ intersect in p; FD and $F'D'$ in q; DE and $D'E'$ in r, then Ap, Bq, Cr are concurrent. (Dr. Mackay, "Symmedians of the Triangle," etc., p. 39.)

80. A', B', C' are the midpoints of the sides of the triangle ABC, and I, I_1, I_2, I_3, are the in and ex centers.

I_1A', I_2B', I_3C' concur at the symmedian point of the triangle $I_1I_2I_3$.

IA', I_3B', I_2C' concur at the symmedian point of the triangle II_3I_2.

I_3A', IB', I_1C' concur at the symmedian point of the triangle I_3II_1.

I_2A, I_1B', IC' concur at the symmedian point of the triangle I_2I_1I.

81. If AK, BK, CK cut the sides of the triangle ABC at the points R, S, T and the circumcircle of the triangle ABC at the points D, E, F, then

AK, BF, CE are concurrent.

BK, CD, AT are concurrent.

CK, AE, BD are concurrent.

82. X, Y, Z are the feet of the perpendiculars in the triangle ABC. If H_1, H_2, H_3 be the ortho-centers of the triangles AYZ, ZBX, XYC, then the lines H_1X, H_2Y, H_3Z are concurrent.

83. If H_1', H_2', H_3' be the ortho-centers of the triangles HYZ, XCZ, XYB.

H_1'', H_2'', H_3'' be the ortho-centers of the triangles CYZ, XHZ, XYA.

H_1''', H_2''', H_3''' be the ortho-centers of the trianglas BYZ, XAZ, XYH.

And if T_1 be the homothetic center of the triangles XYZ and $H_1'H_2'H_3'$.

T_2 be the homothetic center of the triangles XYZ and $H_1''H_2''H_3''$.

T_3 be the homothetic center of the triangles XYZ and $H_1'''H_2'''H_3'''$.

Then AT_1, BT_2, CT_3 concur at the centroid of the triangle XYZ.

(Nos. 80, 81, 82, 83 are extracted from the work of Dr. Mackay in the Proceedings of the Edinburgh Math. Soc.)

84. If through K parallels be drawn to BC, CA, AB, they intersect the corresponding altitudes in A_1, B_1, C_1, respectively, which are the vertices of Brocard's first triangle. BA_1, CB_1, AC_1 concur at Ω; BC_1, CA_1, AB_1 concur at Ω', and thus the two Brocard points are determined.