As an illustration, consider the integral:

$$\int F\,(x,\,y)\;dx.$$

$$y = \sqrt{\left[(x-a)+\sqrt{a\,(x-a)}\right](x-2a)}.$$

Here we have:

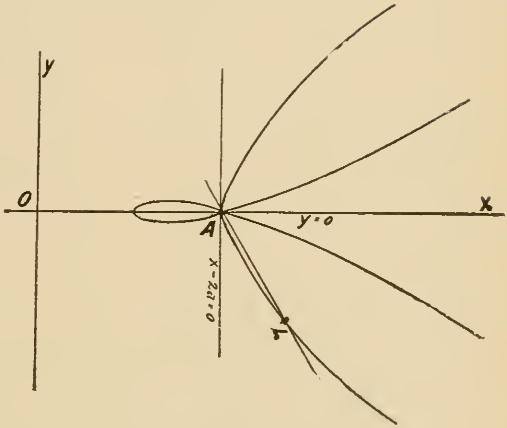
(1.) $f_4 \equiv y^4 - 2y_2\,(x-a)(x-2a)+(x-a)(x-2a)^3 = o,$

and this curve has a triple point at $(2a,\,o)$. Taking A at this point, and

(2.) $y = o,\ x - 2a = o$

as the equations of lines through A, we are to solve for the intersections of $f_4 = o$, and the line:

(3.) $y + \lambda\,(x - 2a) = o.$



Since three solutions are known, we readily find:

(7.) $X\,(x,\lambda) \equiv x\,(\lambda^2-1)^2 - a\,(2\lambda^4 - 2\lambda^2 + 1) = o.$

(8.) $\therefore x = \dfrac{a\,(2\lambda^4 - 2\lambda^2 + 1).}{(\lambda^2-1)^2}$

(9.) $\therefore y = -\lambda\,(x - 2a) = -\dfrac{a\lambda\,(2\lambda^2-1)}{(\lambda^2-1)^2}.$

If the curve $)f_n = o$, instead of having a multiple point of order $(n-1)$, has $\frac{1}{2}\,(n-1)\,(n-2)$ double points, that is, if its deficiency is zero, then it is a unicursal curve, and hence x and y can be expressed rationally in terms of a single parameter, and hence the reduction can be performed.

Alternate Processes. By Professor Arthur S. Hathaway.

I. Introduction.

1. The alternate (and symmetric) processess that we develop seem valuable from their simplicity and power, and their general applicability in all departments of mathematics. They may be employed in any algebra in which addition is associative and commutative without regard to the laws of multiplication.

2. The notation is a doubly dual one, *i. e.*, from a given theorem and proof a dual theorem and proof may be derived by correspondence, and each of these has its dual by another correspondence, so that every theorem is of four-fold interpretation.

3. As illustrative applications we have taken the extensions, to n-fold algebra, of Green's theorem connecting integration through a space with integration over the boundary of that space (the laws of multiplication undetermined); the theory of determinants in any algebra; quaternions, and four-fold space.

4. The alternate processes lead in quaternions to formulas that are almost identical with those of Prof. Shaw's "A Processes," and the two notations are readily convertible. The advantages of our notation are that it pertains to a general theory and that its developments are easy and natural rather than arbitrary and labored.

II. DEFINITIONS.

5. We consider a function, $\phi(p_1, p_2, \ldots p_n)$, of n variables, and substitutions, s, $s,'$, *etc.*, that permute these variables among themselves.

6. We let (s) stand for the assemblage $(s_1, s_2, \ldots s_m)$, (s'), stand for $(s_1', s_2', \ldots s'_m)$, and $(t) = (s)(s')$, stand for $(t_1, t_2, \ldots t_{mm'})$, where $t_u s =_r s'_{r'}$, $r = 1, 2, \ldots m$, $r' = 1, 2, \ldots m'$, and $u = m'(r-1) + r'$, say.

7. We further denote, by $\pm s$, the substitution s, with the factor *1* or -1, according as s involves an even or an odd number of transpositions, and by $e_{(s)}$, the fraction which is the ratio of the excess of the number of positive over the number of negative substitutions in (s) to the whole number of substitutions in (s). When (s) forms a "group" we have $e_{(s)} = 1, -1$, or 0, the latter value in all cases where the group contains both positive and negative substitutions.

8. We denote by $A_{(s)}$, the *alternate process*, $\frac{1}{m} \sum_{r=1}^{r=m} \pm s_r$. This process performed on any operand ϕ before which it is placed, gives as a result a sum of terms, $\Sigma \pm \phi_{s_r}$, divided by the number of terms in the sum, where ϕ_{s_r} is the function ϕ with its variables rearranged by the substitution s_r.

9. When (s) includes all substitutions of the n variables, so that $m = \underline{n}$, the corresponding process is denoted by A. When a process pertains to the group of $\underline{m}$ substitution of m given variables (m not $= n$), it is denoted by $\tilde{A}$ with the affected variables correspondingly marked.

10. A function ϕ is *alternate as to* (s) when $\pm s_r \cdot \phi = \phi$, $r = 1, 2, \ldots m$.

11. A function ϕ is *alternate as to* (s) *for the arrangements* (s') when every $\phi_{s'_{r'}}$ is alternate as to (s).

12. We distinguish between $s_r \cdot \phi = \phi_{s_r}$ and $\cdot\, s_r\, \phi$; *viz.*, the latter function involves the symbol s_r which is a function of the variables so that a substitution on the variables of $\cdot\, s_r \phi$, which have the same order as in ϕ, is not equal to the same substitution on ϕ_{s_r}. In fact $s \cdot s_r\, \phi = s_r\, \phi_s = s_r \cdot s \cdot \phi$.

13. We have also symmetric processes, $C_{(s)}$, *symmetric functions as to* (s), *etc.*, whose definitions are obtained by replacing $\pm s_r$ by s_r in the above definitions. There is a dualty between "alternate" and "symmetric" which consists in the interchange of corresponding terms. The fraction $e_{(s)}$ is in general its own dual.

14. There is also a dual interpretation of the substitution s, *viz.*, write for the moment $\phi\,(p_1, p_2, \cdot \cdot\, p_n) = \phi\,{1,\, 2,\, .\,.\, n \atop 1,\, 2,\, .\,.\, n,}$ where we have the number of a "variable," and beneath it, the number of its "place" in ϕ. Ordinary substitutions affect the upper line of numbers only, *i. e.*, the "variables." The same substitutions on the lower line of numbers only are "place" substitutions. The substitution s that affects the given number *1, 2, . . . n* may be marked $\frown{s}$ or $\underset{\smile}{s}$ according as it affects variable or place numbers. The dualty arises from these two interpretations of the substitutions of any process. When the variables of the operand that are affected by $\overset{\frown}{s}$ occupy the places corresponding to their numbers, we have $\underset{\smile}{s} = \overset{\frown}{s}^{-1}$, and the processes $\tilde{A}_{(s)}$, $\underset{\smile}{A}_{(s)}$ give the same result provided (s) is a substitution group. If, however, the above arrangement of the variables be affected by a substitution s', and the result taken as operand, we have $\underset{\smile}{s} = \overset{\frown}{s'}\, \overset{\frown}{s}^{-1}\, \overset{\frown}{s'}^{-1}$, so that the two processes $\tilde{A}$, $\underset{\smile}{A}$ to the same group (s) are in general different, the latter being equivalent to the former to a group that is similar to (s) only.

III. THEOREMS.

[The proofs are too elementary to need insertion.]

Theor. 1. If ϕ be alternate (or symmetric) as to (s), then is $A_{(s)}\ \phi = \phi$ (or $e_{(s)}\ \phi$).

Theor. 2. If $(t) = (s)\,(s')$, then is $A_{(t)}\ \phi = A_{(s)}\ A_{(s')} \cdot \phi = A_{(s')} \cdot A_{(s)}\ \phi$.

Note.—This result shows that the product of two alternate processes is an alternate process, and that a process $(A_{(t)}$ may be expended in terms of a given minor process $(\,A_{(s)}\,)$.

E. g., $A \cdot p\, q\, r = \frac{1}{3}\,(p\, A\, q\, r - q\, A\, p\, r + r\, A\, p\, q)$, $A \cdot p\, q\, r\, s = \frac{1}{6}\,(A\, p\, q \cdot A\, r\, s + A\, r\, s \cdot A\, p\, q - A\, p\, r \cdot A\, q\, s - A\, q\, s \cdot A\, p\, r + A\, p\, s \cdot A\, q\, r + A\, q\, r \cdot A\, p\, s)$, *etc.*

These are place expansions. Variable expansions give different results,

$e \cdot g.$, $A \cdot p\, q\, r = \frac{1}{3}\,\tilde{A}\,(p\,\hat{q}\ \hat{r} - \hat{q}\, p\ \hat{r} + \hat{q}\ \hat{r}\, p)$

$= \frac{1}{3}\,(p\, A\, q\, r - \tilde{A}\ \hat{q}\, p\ \hat{r} + A\, q\, r \cdot p)$. See art. 16.

Cor. 1. $A_{(t)} \cdot \phi = A_{(s')}\, \phi$ (*or* $e_{(s)} \cdot A_{(s')}\, \phi$, *when* ϕ *is alternate* (*or symmetric*) *as to* (s) *for the arrangemints* (s').

Note.—If (s) be a group this condition means practically for all the arrangements of (t).

Cor. 2. $A_{(t)} \cdot \phi = A_{(s)}\, \phi$ (*or* $e_{(s')} \cdot \phi$) *when* ϕ *is alternate* (*or symmetric*) *as to* (s').

Theor. 3. If (s) *be a group, then* $\cdot\, A_{(s)}\, \phi$ *is an alternate function of the group* (s) *for all arrangements of the variables.*

Note.—$A_{(s)} \cdot \phi$ is an alternate function of the group (s) only for those arrangements $s, \ldots.$ that satisfy $s\,(s) = (s)\,s$. These include the group (s).

Theor. 4. If (s) *be a group, and* (s') *be any assemblage contained in* (s), *then,*

$$A_{(s)}\, \phi = A_{(s)} \cdot A_{(s')}\, \phi = A_{(s)}\, A_{(s')} \cdot \phi\,.\,.$$

15. These are the principal theorems of the subject. We note some important special cases where the processes are those that pertain to all the substitutions of given numbers (variables or places).

16. Let A' affect m' given numbers, let A'' affect m'' other given numbers, and so on. Then $A'\, A'' \ldots$ is a process whose factors are commutative and whose substitutions form a group (s), consisting of substitutions that permute each set of variables (or the variables in each set of places) among themselves. One complementary assemblage (s'), such that $(s)\,(s')$ forms the complete group of $n!$ substitutions then consists of the substitutions that leave each set of variables (or the variables in each set of places) in their original order among themselves. Any element $s'_{r'}$ of this assemblage may be replaced by any product $s_r\, s'_{r'}$ without charging the assemblage as a complement of (s).

We then have from *th. 2.*

Theor. 2'.

$$A\,\phi = A_{(s')} \cdot A'\, A'' \ldots \phi = \frac{m'!\; m''! \ldots}{n!}\, \Sigma \pm s'_{r'} \cdot A'\, A'' \ldots \phi.$$

In this expansion of $A\,\phi$ in terms of minor A's, all terms may be made positive by replacing every negative $s'_{r'}$ by its product by a transposition of (s).

(*a*). $A\,\phi = A_{(s')}\, \phi$ (*or 0*) *when* ϕ *is alternate* (*or symmetric*) *as to* (s) *for all arrangements of the variables.*

Note.—In particular, *if* ϕ *be symmetric as to certain variables* (*or places*) *for all arrangements of the variables, then* $A\,\phi = 0$.

(*b*). $A\,\phi = A'\, A'' \ldots \phi$ (*or* $e_{(s')} \cdot A'\, A'' \ldots \phi$) *when* ϕ *is alternate* (*or symmetric*) *as to* (s').

IV. LINEAR ALTERNATES.

17. A function ϕp is said to be *linear* when $\phi\,(x\,p + y\,q) = x\,\phi\,p + y\,\phi\,q$, where x, y are ordinary numbers (scalars).

18. A function $\phi\,(p_1, p_2, \ldots p_m)$ is a *linear alternate of* m^{th} *order*, when it is linear as to each of its variables, and the interchange of any two variables changes its sign.

Theor. 5. A linear alternate vanishes when one variable is zero, or two variables are equal. It is unaltered by adding to any variable any sum of scalar multiples of the remaining variables. It vanishes when two or more of its variables are linearly dependent —in particular, when the order of the alternate is greater than the order of the algebra.

19. It is easily seen that in an algebra of n'^{th} order the general linear alternate of m'^{th} order is a sum of algebraic multiples of $\underline{|n} \,/\, \underline{|m}\ \underline{|n-m}$ independent scalar alternates of m'^{th} order.

20. If $\phi\,(p_1, p_2, \ldots p_m)$ be a linear function of m'^{th} order, then by *th 3* and note, $\cdot A\,\phi$ and $A\,\cdot\,\phi$ are linear alternates of that order. Also we have more constants than we need (n^m) in order to make either of these the most general linear alternate of m'^{th} order; in fact we have more than enough constants to make also $\cdot\,C\,\phi$ or $C\,\cdot\,\phi$ the most general linear symmetric of m'^{th} order.

21. In the use of $A_{(s)}$ it is not only well to note that it is a linear symbol, but also that it is commutative with any constant linear symbol, ψ, of one variable (such as S, V, K, in quaternions). In applying A, however, to a function ϕ we can not reduce the value of ϕ by reason of any special values of the variables *i, e,* if for special values of the variables we have $\phi = \phi'$, we do not therefore have $A\,\phi = \phi'$.

22. In any algebra of n'^{th} order, we may take the units $i_1, i_2, \ldots i_n$ as the numbers of n independent directions of unit length (not necessary rectangular). Also, any number $p = x_1\, i_1 + x_2\, i_2 + \ldots + x_n\, i_n$ where $x_1, x_2, \ldots x_n$, are ordinary numbers) may be taken as the number of a line whose components, according to the parallelogram law of addition, are $x_1\, i_1,\ x_2, i_2, \ldots x_n\, i_n$. Taking a fixed origin O, any point P has a definite co-ordinate p, the number of the line $O\,P$. Any number of independent lines have two orders of arrangement such that the interchange of two lines changes the order of arrangement. A change of order in the argument lines of an alternate therefore changes its sign.

23. Consider an m-space bounded by the tangential paths of m independent differentials $d_1\, p,\ d_2\, p, \ldots d_m\, p$. This space may be taken so small as to be approximately an m-parallelogram whose r^{th} pair of opposite faces intersect the lines of $d_r\, p$ and contain the remaining lines through the points of these faces. By $(r - 1)$ interchanges the "r'^{th}" order $d_r\, p,\ d_1\, p, \ldots d_{r-1}\, p,\ d_{r+1}\, p, \ldots d_m\, p$ becomes the "1^{st}" order $d_1\, p, \ldots d_m\, p$. These interchanges may be made so as to leave $d_r\, p$ first. At the initial "r^{th}" face $d_r\, p$ is inward, and we have r interchanges from the r'^{th} order in the differentials exclusive of $d_r\, p$ to bring the r'^{th} order to the first order, say $d'_1\, p,\ d'_r\, p,\ d'_m\, p$, wher $d'_1\, p = -\, d_r\, p$ is *outward.*

24. If $\varphi(p, d_1 p, d_2 p, \ldots d_{r-1} p, d_{r+1} p, \ldots d_m p) = \varphi_r$, be any function of p, that is linear in the differentials, then if d_r denote differentiation corresponding to $d_r p$, while A affects the subscripts of the differential symbols that follow it, we have for integration through the m space,

$$(-1)^{r-1} \int^m d_r \cdot A \varphi_r = \int^{m-1} A \varphi(p, d'_2 p, \ldots d'_m p)$$

over the r'^{th} pair of faces, where $d'_1 p = \pm d_r p$ is *outward* at each face. Thus summing for every pair of faces and noting that, by *th 2′*, $\Sigma (-1)^{r-1} d_r A \phi_r = m A \,.\, d_1 \phi(p, d_2 p, d_3 p, \ldots d_m p)$, we have the following theorem connecting integration through any m-space with integration over the (m-1) boundary of that space:

Theor. 7.

$$m \int^m A \,.\, d_1 \varphi(p, d_2 p, d_3 p, \ldots d_m p)$$
$$= \int^{m-1} A \varphi(p, d'_2 p, d'_3 p, \ldots d'_m p).$$

25. The elements $d_1 p, \ldots d_m p$ are in the same order of arrangement throughout the integration; and at the boundary $d'_1 p, \ldots d'_m p$ are in the same order, with $d'_1 p$ outward.

Any space may be divided into these small finite spaces by the differential lines and the contributions to the boundary integral made by intermediate faces cancel each other, except where the value of φ is different on two sides of the same face at the same point of it. In this case such intermediate boundary must be retained in computing the boundary integral.

V. ALTERNATE PRODUCTS.

26. Let $\varphi(p_1, p_2, \ldots p_n) = \varphi_1 p_1 \,.\, \varphi_2 p_2 \, \varphi_n p_n$.

We then have:

$$n \,A \,.\, \varphi = \begin{vmatrix} \varphi_1 p_1, & \varphi_2 p_1, & \ldots & \varphi_n p_1 \\ \phi_1 p_2 & \phi_2, p_2, & \ldots & \varphi_n p_2 \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_1 p_n & \varphi_2 p_n & \ldots & \varphi_n p_n \end{vmatrix}$$

27. This determinant is multiplied out by ordinary rules, except that since the factors may not be commutative, or even associative, each product must appear in the order and association given by φ. The expansions of *Theor. 2′* are then expansions of this determinant in terms of its minors either by rows or columns, according as we employ place or variable substitutions.

28. When two factors of the product φ are commutative, for all arrangements of the variables, such factors are also commutative in $A \varphi$. The interchange of the corresponding functional symbols alone has, therefore, the same

effect in changing the sign of $A\,\phi$ as the interchange of the variables alone. Hence when all the factors are commutative, the A may operate either on the variables or on the functional symbols, and since $A\,\phi$ is linear in the latter, it has the properties of linear alternates with respect to the functional symbols. If the functional symbols be linear, the alternate $A\,\phi$ is also linear with respect to the variables.

VI. QUATERNIONS.

29. We consider linear alternate products whose functional symbols are *1*, S, V, K. The symbol S gives a factor that is commutative with any other factor, so that any other symbol in the same product with S may be reduced by $\pm n\, S$, where n is a scalar.

30. By substituting $1 = S + V$, $K = S - V$ and expanding, our linear alternate product of any order is found to depend on two in which the symbols are either all V or one S and the rest V. Two S symbols give an alternate product that is identically zero (*Theor. 2, note*). It appears that: the two of second order are vectors; the two of third order are a scalar and a vector; one of the fourth order is zero, the other is a scalar. Any linear alternate of fifth or higher order is identically zero.

31. In the geometrical interpretation in which *1*, i, j, k are the numbers of four mutually perpendicular unit lines in four-fold space, the condition of perpendicularity of p, q is $S\,.p\,K\,q = o = S\,.\,K\,p\,.\,q$, *i e.*, $p\,K\,q = -q\,K\,p$, $K\,p\,.\,q = -K\,q\,.\,p$. Thus in any alternate product whose functional symbols are alternately *1*, K, and whose variables occur in sets, such that any two of different sets are perpendicular, we have

$$A\,\phi = A'\,A'' \ldots \phi,$$

where $A'\,A''$ are alternate symbols that affect the different sets of variables [*th 2′* (*b*), *art. 16.*] In particular, if all the variables are mutually perpendicular, then $A' = 1$, $A'' = 1$, and $A\,\phi = \phi$.

32. The alternates of second order are:

(a). $A\,.\,Vp\,.\,Vq = V\,.\,Vp\,.\,Vq = L\,i + jM + N\,k.$

(b). $2\,A\,.\,S\,p\,.\,V\,q\,. = 2\,A\,.\,S\,p\,.\,q = A\,i + B\,j + C\,k.$

A, B, C, L, M, N are the six independent scalar linear alternates of second order, and are the coefficients of $\phi\,(1, i)$, $\phi\,(1, j)$, $\phi\,(1, k)$, $\phi\,(j, k)$, $\phi\,(k, i)$, $\phi\,(i, j)$, in the expansion of any linear alternate $\phi\,(p, q)$.

33. We have further:

(a). $A . pq = A . V . pq = A . Vp . Vq = V . Vp\ Vq = A . Kp . Kq.$

(b). $A . p\ Kq = V . p\ Kq = -2\ A . Sp . q - A . pq.$

(c). $A . Kp . q = V . Kp . q = 2\ A . Sp . q - A . pq.$

Note.—These and similar formulas are useful in computing alternates of higher order. Thus a factor pq of a product may be replaced by $A.\ pg$ [*th 4*] or any of its equivalent values in (a) with or without the partial A.

34. Resolve q into $q' + q''$ respectively parallel and perpendicular to p. Then $A . p\,Kq = A . p\ Kq'' = p .\ Kq''$ [*th 6, art 31*$_1$]. Its tensor is therefore *base* $\times$ *altitude* of parallelogram on q, p, as sides. We call A . p K q the *vector* area of the parallelogram (q, p). It gives plane, direction and tensor, by the plane, direction of turn, and tensor of the vector. Observe that $A . p\,q = V . Vp\ Vq$ is perpendicular to the three-space (1, Vp, Vq) $=$ (1, p, q).

35. The alternates of third order are:

(a). $A . Vp . Vq . Vr = A . \breve{A} . V\hat{p} . Vq . V\hat{r} = A .\ S.\ V\,p .\ Vq\ Vr =$ $S\ Vp . Vq . Vr.$ We call this scalar $- a$.

(b). $3\ A . Sp . qr = 3\ A . Sp . Vqr = b\,i + c\,j + d\,k.$

The four independent scalar alternates of third order are a, b, c, d, respectively the coefficients of $\circ$ (i, j, k), $\circ$ (1, j, k), $\circ$ (1, k, i) $\circ$ (1, i, j) in the expansion of any linear alternate ϕ (p, q, r).

36. We have further:

(a). $A . pq\,r = A . V\,p . q\,r + A . S\,p . q\,r$

(b). $S . A . pq\,r = S . V\,p . V\,q . V\,r = S . p\,A\,qr = S\,A . p\,A\,q\,r =$ $\frac{1}{3}\,S\,(p\,A\,q\,r + q\,A\,r\,p + r\,A\,p\,q)$ *etc.*

(c). $V . A . p\,q\,r = A . S . p . q\,r = -A . p . S\,q . r$ *etc.*, $=$ $\frac{1}{3}\,V . (p . A\,q\,r + q\,A\,r\,p + r\,A\,p\,q)$ *etc.*

(d). $A . p . Kq . r = -S . A\,pq\,r = 3 . V . A\,pq\,r$

(e). $A . Kp . q . Kr = -KA . p . Kq . r = SA . pq\,r - 3\,V . A\,pq\,r$ $= S . p\,A\,q\,r - 3\,A . S\,p . A\,q\,r.$

Note.—This alternate is Shaw's A . p q r, and his formulas hold in the present notation with this value of his $A . pqr$. In the present notation a function that is used as a variable must be enclosed in brackets. Thus A [S p] q = 0, where the S follows the p, but $A . Sp . q$ is not zero. Similarly, Shaw's value of $A . pq . A\,rst$ becomes $A . Kp . q . K[A . Kr . S . Kt] = 6\,\breve{A} . \hat{r} . Sp\hat{s} . Sq\hat{t}.$

37. Resolve r into $r' + r''$ respectively parallel and perpendicular to the plane of p, q, and then $A . p . Kq . r = (A . p . Kq) . r''$, whose tensor is *base* $\times$ *altitude* of the parallelopiped on p, q, r as edges. We call this the quaternion volume of parallelopiped. It will be shown that this is a line perpendicular to the edges

of the parallelopiped in the relative direction of 1 to i, j, k. We have $A . p . Kq . 1 = -3 A p q$. Also $A . Kp . q . Kr$ is the quaternion volume of the parallelopiped $(Kp, Kq, Kr.)$

38. The alternates of fourth order are:

(a). $A . Vp . Vq . Vr . Vs . = 0$, since it is its own conjugate and is in form the vector $A . Vp . S . Vq . Vr . Vs$.

(b). $4 A . Sp . Vq\, Vr\, Vs = 4 A . Sp . S . Vq . Vr . Vs = -D$, a scalar. D is the coefficient of $\phi\,(1, i, j, k)$ in any linear alternate $\phi\,(p, q, r, s)$. All our alternates of fourth order are scalars, zero when they can be shown formally as vectors.

39. We have further:

(a). $0 = A . p q r s = A . Sp . V . q r s = A . Vp\, S . q r s$, *etc.*

(b.) $S . p . A . Kq . r\, Ks = 4 A . Sp . S . q\, A . r s$

$= -4 A . Vp . V . q\, A\, r s = 4 A . Sp . Vp\, Vq . Vs$. *etc.*,

$= A . S . p . Kq . r . Ks = A . p . Kq . r . Ks$. *etc.*

Note.—The first equation of (b) follows from *36 e*, thus: $A . Kq . r . Ks = S . q\, A\, r s - (Sq . A\, r s - Sr . A . q s + Ss . A . q r)$, and operating by $S . p$ we find (b).

Note.—From (b), $S . s\, A . Kp . q\, Kr = - S . s\, K A\, p . Kq . r$ is an alternate of fourth order; it therefore vanishes when $s = p, q$, or r, — in other words $A\, p . Kq . r$ is perpendicular to the lines p, q, r. To find the order in space make $p, q, r = i, j, k$, whence $A . p . Kq . r = i . Kj . k = 1$.

(c). $S . Kp\, A . q . Kr . s = - S . p\, A . Kq . r\, Ks = D = A . Kp . q\, Kr . s$, *etc.*

40. Resolve p into $p' + p''$ respectively in and perpendicular to the space q, r, s, thus:

$$A . Kp . q . Kr . s = Kp'' . A\, q\, Kr\, s,$$

whose tensor is *altitude* $\times$ *base* of the four parallelogram on p, q, r, s as sides. This is the scalar content of the four-parallelogram, positive when in the order $1, i, j, k$ since, substituting these values in the alternates, the result is $K1 . i . Kj . k = 1$.

41. We have identically $A . Kp . q . Kr . s . t = 0$.

Let $p' = A . q . Kr . s$, $q' = - A . p . Kr . s$,

$r' = A . p . Kq . s$, $s' = - A . p . Kq . r$

Then $A . Kp . q . Kr . s = Sp'\, Kp = Sq'\, Kq$, *etc.*,

and our expanded identity is,

(*a.*) $t\, Sp'\, Kp = p\, S . t\, Kp' + q\, S . t\, Kq' + r\, S . t\, Kr' + s . St\, Ks'$.

42. If $\phi\, s$ be a linear function, we have

(*a.*) $4 A . p . Kq . r . \phi s = c . Sp'\, Kp$, where $c = 4 A . 1 . Ki . j . \phi k$

$= -[\phi 1 + i \phi i + j \phi j + k \phi k]$.

We have $c = -t$ when $\phi s = S t K s$, and (a) becomes,

(b.) $t S p' K p = p' S t K p + q' S t K q + r' S t K r + s' S t K s$.

43. If $\psi(p, q)$, $\phi(r, s)$ be linear functions, we have $6 . A . \psi(p, q) . \phi(r, s) = c S p' K p$ where $c = A \psi(1, i) . A \phi(j, k) + A \psi(j, k) . A \phi(1, i) +$ two similar terms found by advancing i, j, k.

If $\psi(p, q) = \psi_1(p K q)$, then $A \psi(1, i) = A \psi j k = - \psi_1 i$, *etc.*

" $= \psi_1(K p q)$, then $A \psi(1, i) = - A \psi(j, k) = \psi_1 i$, *etc.*

" $= \psi_1(p q)$, then $A \psi(1, i) = 0$ $A \psi(j k) = \psi_1 i$, *etc.*

" $= 2 \psi_1(S p . q)$ then $A \psi_1 i = \psi_1 i$, $A \psi(1 k) = 0$, *etc.*

We have thus the identities,

(a.) $A . \psi(p K q) \phi(K r s) = 0$.

(b.) $A . \psi(p q) \phi(r s) = 0$.

(c.) $A . \psi(S p . q) \phi(S r . s) = 0$.

(d.) $6 . A . p q . S(r K r_1) S(s K s_1) = A_1 S r_1 . s_1 . S p' K p$.

In fact these methods may be employed to multiply formulas indefinitely. The above are interesting as giving the general relations between the six vector alternates of the same form that may be derived from the quaternions p, q, r, s.

44. We note the following geometrical interpretations: (p_1, q_1, *etc. not affected by* A).

(a.) $2 A . p S . q K q_1$ *is a line in the plane* (p, q) *that is perpendicular to* q_1; *viz., it is* $A p K q$. *projection of* q_1 *on the plane* (p, q).

That it is a line perpendicular to q_1 in the plane (p, q) is seen by its form and the fact that the operator $S . K q_1$ gives an alternate of a symmetric product which is zero by *th* 2'.

Note.—We have, for the complete proof:

$$\widehat{A} . \widehat{p} . K \widehat{q} . \widehat{q_1} = \tfrac{1}{3} A (p . K q . q_1 - p . K q_1 . q + q_1 . K p . q)$$

$$2 A . p S . q K q_1 = A (p . K . q_1 . q + p . K q q_1) = A (q K q_1 . p + q_1 K q . p) = \tfrac{1}{3} A (2 p . K q . q_1 + p . K q_1 . q - q_1 . K p . q),$$

so that $\widehat{A} . \widehat{p} . K \widehat{q} . \widehat{q_1} . + 2 A . p S . q K q_1 = A p K q . q_1$. In this result resolve q_1 into $q_1'' + q_1'''$ respectively parallel and perpendicular to the plane (p, q) and it becomes (since $\widehat{A} . \widehat{p} . K \widehat{q} . \widehat{q_1} = A p K q . q_1'''$),

$$2 A . p . S . q K q_1 = A . p K q . q_1'' \quad Q. E. D.$$

45. Operating on the last result by $S . K p_1$, and remembering, since the planes (p, q), (p_1, q_1''') are perpendicular, that $S . A p K q . A_1 p_1 K q_1'' = o$, we find,

(a). $2 A . S(p K p_1) S(q K q_1) = - S . A p K q . A_1 p_1 K q_1$

$=$ product of areas times cosine of angle between the planes of the parallelograms (p, q), (p_1, q_1). If we drop the subscripts after expansion, we have the squared tensor of the area of (p, q) *viz.*, $T^2 A p K q$.

46. Similarly we have,

(a). $6\,A\,.\,p\,.\,S\,(q\,K\,q_1)\,S\,(r\,K\,r_1) =$ *line in space* (p, q, r) *perpendicular to plane* (q_1, r_1).

This line is therefore $-A_1\,.\,q_1\,.\,K r_1\,.\,[\,A\,.\,p\,.\,K q\,.\,r\,]_1$, the constant factor being determined by putting $p, q, r = i, j, k$, $q_1, r_1 = j, k$. This beeomes, to the factor $S\,r\,K\,r_1$, the line of *44* (a) when r_1 is perpendicular to the plane (p, q)

(b). $6\,A\,.\,S(\,p\,K p_1\,)\,S(\,q\,K q_1\,)\,S(\,r\,K r_1\,)$

$= S\,.\,A\,.\,p\,.\,K q\,.\,r\,K\,.\,A_1\,.\,p_1\,.\,K q_1\,.\,r_1$

$=$ product of volumes times cosine of angle between the spaces of the parallelopipeds (p, q, r), (p_1, q_1, r_1)

(c). $24\,A\,.\,p\,S(q\,\,K\,q_1\,)\,S\,\,(\,r\,K r_1)\,S\,(s\,K\,s_1\,) =$ *line perpendicular* to $(p_1, q_1, r_1) = A_1\,p_1\,.\,\,K q_1\,.\,r_1\,.\,S p'\,K p$.

This becomes, to the factor $S\,(s\,K s_1)$, the line (a) when $s_1 \perp (p, q, r)$.

(d). $24\,A\,.\,S(p\,K p_1)\,S(q\,K q_1)\,S(r\,K r_1),\,S\,(s\,K s_1)$

$= S\,p'\,K\,p\,.\,S\,p'_1\,K\,p_1 =$ product of scalar contents of (p, q, r, s), $(\,p_1, q_1, r_1, s_1)$.

47. We have given sufficient illustrations of the value of alternate processes. The symmetric processes are capable of similar development although we have scarcely touched upon them.

A New Form of Galvanometer. By J. Henry Lendi.

The galvanometer which I am about to describe is a result of the difficulties experienced in attempting to make use of several very sensitive galvanometers in the physical laboratory of the Rose Polytechnic Institute. These difficulties are due to local changes in the earth's magnetic field, arising from moving locomotives, electric motors and street cars in the neighborhood of the laboratory. It will be seen that the existing conditions are anything but favorable to the use of a very sensitive galvanometer depending on the earth's field for the directive force.

In the past year or two several attempts have been made to overcome this difficulty by making a galvanometer of the D'Arsonval type; that is, one in which the directive force is independent of the earth's field. This galvanometer differed only from the ordinary D'Arsonval instrument in that the field was excited by an auxiliary battery instead of a permanent magnet. By this means we were able to secure a very intense controlling field, and thereby, thought we should be able to make a galvanometer