On the Reduction of Irrational Algebraic Integrals to Rational Algebraic Integrals. By John B. Faught.

The Inverse operations of Analysis are more interesting and fruitful than the Direct, since each demands a new field of quantity or a new kind of function in order that it may be possible without exception. Thus negative numbers have their origin in subtraction, fractions grow out of division, and irrational and imaginary numbers arise in the extraction of roots. The same thing is true of integration considered as the inverse of differentiation.

At the time of the discovery of the Calculus the algebraic and certain elementary transcendental functions were known. The algebraic functions included all those expressions which can be formed by a finite combination of the processes of addition, subtraction, multiplication, division, involution and the extraction of roots. The transcendental functions included the exponential, logarithmic, trigonometric and circular functions. These will be called the elementary functions, and exclude the infinite series.

It is a fundamental theorem of the Integral Calculus that the integral of any rational algebraic function can be expressed in terms of the elementary functions. This is sometimes expressed by saying that any rational algebraic function can be integrated.

The attention of mathematicians was early directed to those integrals that are made irrational by the presence of the square or other root of a polynomial of the first, second and higher degrees. It was soon found that if the irrationality was due to a square root of a polynomial of the first or second degree the integral could be expressed in terms of the elementary function. The integration being accomplished in each case by reducing the irrational function to a rational function and then performing the integration. This method, however, was found to fail, in general, as soon as the polynomial under the radical is of the third or higher degree. The investigation of irrational algebraic integrals led to the discovery of the Abelian functions of which the Hyperelliptic and Elliptic functions are special cases. By means of these functions the integral of any algebraic function can be expressed.

The integrals under consideration here are known as Abelian Integrals and are defined thus:

$$\int F(x, y)\, dx.$$

where y is defined by:

$$f_n(x, y) = o,$$

and F denotes a rational function of x and y.

If y is expressed as an explicit function of x, the expression will contain, in general, a root of some polynomial. The definition is sometimes stated as follows:

$$\int F(x, y)\, dx,$$
$$y = \sqrt[m]{R_n(x)}.$$

Byerly (Integral Calculus Ch. VI) observes that "very few forms (of the above type) are integrable, and most of these have to be rationalized by ingenious substitutions." It is the purpose of this paper to determine the conditions under which irrational algebraic integrals can be reduced to rational algebraic integrals and to present a method of obtaining a substitution by which the integral is rationalized.

Given then, the integral:

$$\int F(x, y)\, dx.$$
$$f_n(x, y) = o,$$

where F is a rational function, to determine the conditions under which this integral can be reduced to the integral:

$$\int r(\lambda)\, d\lambda.$$

where r is a rational function.

Let us consider in the first place the integral:

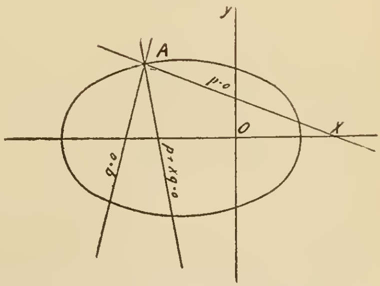
$$\int F(x, y)\, dx.$$
$$y = \sqrt{R(x)}$$

where R (x) is a polynomial of the second degree.

The equation:

(1.) $y^2 - R(x) = o$

is of the second degree and hence represents a conic section. Let A be any point on the curve and let



(2.) $p = o,\ q = o,$

be the equations of any two lines through A. Then

(3.) $p + \lambda q = o,$

represents any line L through A. Since this equation is linear in x and y it can be solved for y by rational processes. Let the solution be:

(4.) $$y = \phi(x, \lambda).$$

(5.) $$\therefore y^2 = \phi^2(x, \lambda).$$

(6.) $$\therefore \phi^2(x, \lambda) - R(x) = o.$$

This is the equation for the determination of the points of intersection of L and the conic. One solution is known, viz.: $x = x_a$. Hence $x - x_a$ is a factor. Divide by $x - x_a$ and call the resulting equation:

(7.) $$X(x, \lambda) = o.$$

This equation is linear in x and therefore can be solved for x by rational processes. Let the solution be:

(8.) $$x = \psi(\lambda).$$

(9.) $$\therefore y = \varphi[\psi(\lambda)].$$

(10.) $$dx = \psi'(\lambda)\, d\lambda.$$

(11.) $$\therefore \int F(x, y)\, dx = \int F \left\{ \psi(\lambda), \varphi[\psi(\lambda)] \right\} \psi'(\lambda)\, d\lambda.$$

Since F, φ, ψ, ψ' are all rational functions, it follows that

$$F \left\{ \psi(\lambda), \varphi[\psi(\lambda)] \right\} \psi'(\lambda).$$

is a rational function of λ. Call this function r (λ).

(12.) $$\therefore \int F(x, y)\, dx = \int r(\lambda)\, d\lambda.$$

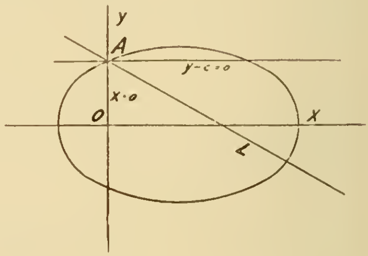
As an illustration consider the integral:

$$\int F(x, y)\, dx,$$

$$y = \sqrt{ax^2 + bx + c^2}.$$

(1.) $$y^2 = ax^2 + bx + c^2.$$

Take A as the point (o, c)



and as the lines through A:

(2.) $$y = c,\ x = o.$$

Then the equation of L is:

(3.) $$y - c + \lambda x = o.$$

(4.) $$\therefore y = \varphi(x, \lambda) = c - \lambda x.$$

(5.) $$\therefore y^2 = (c - \lambda x)^2.$$

(6.) $$(c - \lambda x)^2 - (ax^2 + bx + c^2) = o.$$

(7.) $$\therefore X(x, \lambda) \equiv (\lambda^2 - a)\, x - (2c\lambda + b) = o.$$

(8.) $$\therefore x = \psi(\lambda) = \frac{2c\lambda + b}{\lambda^2 - a}$$

(9.) $$y = \varphi[\psi(\lambda)] = \frac{c\lambda^2 + b\lambda + ac}{a - \lambda^2}.$$

(10.) $$dx = \psi'(\lambda)\, d\lambda = 2\, \frac{c\lambda^2 + b\lambda + ca}{(a - \lambda^2)^2}\, d\lambda.$$

(11.) $$\int F(x, y)\, dx = 2\int F\left(\frac{2c\lambda + b}{\lambda^2 - a}, \frac{c\lambda^2 + b\lambda + ca}{a - \lambda^2}\right) \cdot \frac{c\lambda^2 + b\lambda + ca}{(a - \lambda^2)^2}\, d\lambda.$$

(12.) $$\therefore \int F(x, y)\, dx = \int r(\lambda)\, d\lambda.$$

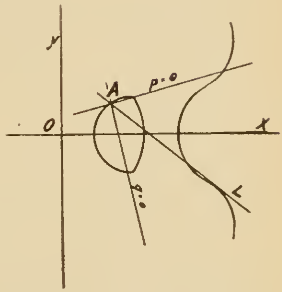
In particular:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c^2}} = -2\int \frac{d\lambda}{a - \lambda^2}.$$

This method always furnishes the substitution by which the reduction is effected, viz.: $\chi = \frac{2c\lambda + b}{\lambda^2 - a}$.

Consider next the integral:

$$\int F(x, y)\, dx$$
$$y = \sqrt{ax^3 + bx^2 + cx + d}.$$



(1.) $$y^2 = ax^3 + bx^2 + cx + d.$$

(2.) $$p = o,\ q = o.$$

(3.) $$p + \lambda q = o.$$

(4.) $$y = \phi\,(x, \lambda)$$

(5.) $$\phi^2\,(x, \lambda) - (ax^3 + bx^2 + cx + \phi) = o.$$

(6.) $$X\,(x, \lambda) \equiv \frac{\phi^2\,(x, \lambda) - (ax^3 + b\,x^2 + cx + d)}{x - x_a} = o.$$

Now the equation:

$$X\,(x, \lambda) = o.$$

is of the second degree in x and hence can not in general be solved by rational processes. The necessary and sufficient condition for the reduction of the given integral to a rational integral is that the coordinates of at least one point of intersection of L and the cubic:

$$f_3 \equiv y^2 - (ax^3 + bx^2 + cx + d) = o.$$

be expressible rationally in terms of λ. This will certainly be true if the cubic, $f_3 = o$, has a double point. For taking A at the double point two solutions of the equation:

$$\phi^2\,(x\lambda) - (ax^3 + bx^2 + cx + d) = o.$$

are known, and, after dividing by $(x - x_a)^2$, the equation:

$$X\,(x, \lambda) = o.$$

is a linear equation, and can be solved by rational processes.

If now the cubic, $f_3 = o$, has a double point, then the first polar:

$$D f_3 = o^*$$

that is: $\frac{d f_3}{d x'} = o, \; \frac{d f_3}{d y'} = o, \; \frac{d f_3}{d z'} = o,$

where f_3 has been made homogeneous by the introduction of the variable z.

If $f_3 = o$ has a double point then the curve is a unicursal or rational curve. Indeed, if one solution of $X(x, \lambda) = o$ is rational, the other must also be rational, and hence the cubic, $f_3' = o$, is unicursal, and hence it must have a double point, since its deficiency is zero.

Theorem: The integral:

$$\int \frac{F(x, y)\, dx}{y = \sqrt{R_3(x)}},$$

can be reduced to a rational integral only when the cubic:

$$f_3 = y^2 - R_3(x) = o$$

is a unicursal curve.

Consider next the general integral:

$$\int \frac{F(x, y)\, dx}{y = \sqrt[m]{R_n(x)}}, \; m \leq n.$$

The curve:

$$f_n = y^m - R_n(x) = o$$

is of the n[th] order, and the equation:

$$X(x, \lambda) = o$$

is of the (n-1)[st] degree. If one rational solution of this equation can be found the reduction can be made, otherwise not.

Suppose the curve: $f_n = o$, has a multiple point of order k, then, by taking A at this point, the equation

$$X(x, \lambda) = o$$

is of degree (n-k). In this case it is necessary to find a rational solution of an equation of the (n-k)[th] degree. Now $f_n = o$, has a multiple point of orner k if the (k-1)[st] polar of that point vanishes identically.

$$D^{k-1} f_n = o.$$

If $f_n = o$ has a multiple point of order (n-1), that is if:

$$D^{n-2} f_n = o,$$

then the equation:

$$X(x, \lambda) = o,$$

is linear and the reduction is always possible.

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* Clebsch. Vorlesungen über Geometrie. Vol. I, p. 315.

As an illustration, consider the integral:

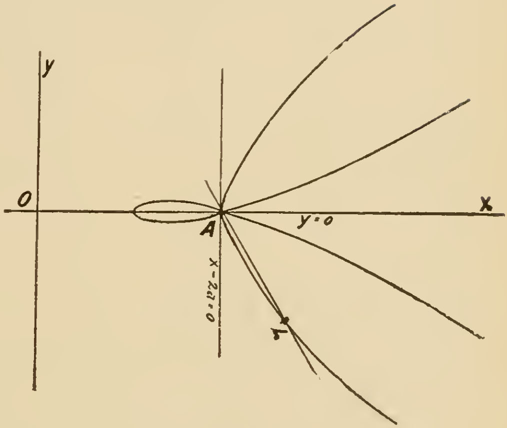
$$\int F\,(x, y)\,dx.$$

$$y = \sqrt{\left[(x-a) + \sqrt{a\,(x-a)}\right](x-2a)}.$$

Here we have:

(1.) $f_4 \equiv y^4 - 2y_2\,(x-a)(x-2a) + (x-a)(x-2a)^3 = o,$

and this curve has a triple point at $(2a, o)$. Taking A at this point, and



(2.) $y = o,\ x - 2a = o$

as the equations of lines through A, we are to solve for the intersections of $f_4 = o$, and the line:

(3.) $y + \lambda\,(x - 2a) = o.$

Since three solutions are known, we readily find:

(7.) $X\,(x, \lambda) \equiv x\,(\lambda^2 - 1)^2 - a\,(2\lambda^4 - 2\lambda^2 + 1) = o.$

(8.) $\therefore x = \dfrac{a\,(2\lambda^4 - 2\lambda^2 + 1).}{(\lambda^2 - 1)^2}$

(9.) $\therefore y = -\lambda\,(x - 2a) = -\dfrac{a\lambda\,(2\lambda^2 - 1)}{(\lambda^2 - 1)^2}.$

If the curve $)f_n = o$, instead of having a multiple point of order $(n-1)$, has $\frac{1}{2}(n-1)(n-2)$ double points, that is, if its deficiency is zero, then it is a unicursal curve, and hence x and y can be expressed rationally in terms of a single parameter, and hence the reduction can be performed.

Alternate Processes. By Professor Arthur S. Hathaway.

I. Introduction.

1. The alternate (and symmetric) procesess that we develop seem valuable from their simplicity and power, and their general applicability in all departments of mathematics. They may be employed in any algebra in which addition is associative and commutative without regard to the laws of multiplication.