Collinear Sets of Three Points Connected With the Triangle. By Robert J. Aley.

This paper does not claim to be either original or complete. It contains a fairly complete list of collinear sets connected with the triangle. All cases of collinearity connected with polygons of more than three sides have been omitted.

The subject of collinearity is both interesting and fruitful. There are three well defined methods of proving the collinearity of three points. The classic one is the application of the theorem of Menelaus: "If D, E, F are points on the sides B C, C A, AB respectfully of AB C, such that $BD \times CE \times AF = - DC \times EA \times FB$, then D, E, F are collinear." In many cases the data are insufficient for the use of this method. Another method of frequent use is to prove that the angle formed by the three points is a straight angle. The author has used another method, believed to be original with him, when the points in question are such that the ratios of their distances from the sides can be determined. This method is fully illustrated in "Contributions to the Geometry of the Triangle."

Collinear problems fall into two very well marked classes. The first class is made up of those points which are definitely located with respect to the triangle. The second class is made up of those points which are located with reference to some auxiliary point.

NOTATION.

In order to save time in the enunciation of propositions the following notation will be used:

A B C is the fundamental triangle.

$A_1 B_1 C_1$ is Brocard's first triangle.

$M_a M_b M_c$ is the triangle formed by joining the middle points of the sides of AB C.

$M_{1a} M_{1b} M_{1c}$ is the triangle formed by joining the middle points of the sides of $A_1 B_1 C_1$.

M is the centre of the circumcircle of AB C.

M^1 is the point isotomic conjugate to M.

G is the median point or Centroid of AB C.

K is Grebe's point or the Symmedian point.

D is the centre of perspective of AB C and $A_1 B_1 C_1$.

D^1 is the point isogonal conjugate to D.

H is the Orthocentre.

Ω and Ω^1 are the two Brocard points.

N is Tarry's point.

Q is Nagel's point. (It is the point of concurrency of the three lines joining the vertices to the points of tangency of the three escribed circles.)

Q^1 is the isotomic conjugate of Q.

O is the centre of the inscribed circle.

S is the point of perspective of M_a M_b M_c and M_{1a} M_{1b} M_{1c}.

S^1 is the point isogonal conjugate to S.

R is the point of concurrence of perpendiculars from A, B, C on the sides of Nagel's triangle.

M_1 is the centre of Nagel's circle.

T is the point of concurrence of perpendiculars from A′ B′ C′ upon the respective sides of A″ B″ C″.

Z is Brocard's centre.

Z^1 is the point isogonal conjugate to Z.

P is the point isotomic conjugate to O.

P^1 is the point isogonal conjugate to P.

Q_1 is the point isogonal conjugate to Q^1.

F is the centre of Nine points circle.

THEOREMS.

The original sources of the theorems are known in only a very few cases. The references simply indicate where the theorems may be found.

(1.) M, H and G are collinear.

(Lachlan—Modern Pure Geometry, p. 67.)

(2.) K, G and the Symmedian point of M_a M_b M_c are collinear.

(Ibid, p. 138.)

(3.) Tangents to the circumcircle at the vertices of AB C form the triange P Q R; H_a, H_b, H_c are the feet of the altitudes of AB C; P H_a, QH_b, RH_c are concurrent in a point which is collinear with M and H.

(Ibid, p. 138.)

(4.) M, K and the orthocentre of its pedal triangle are collinear.

(McClellan—The Geometry of the Circle, p. 83.)

(5) M and the orthocentre of its pedal triangle are equidistant from and collinear, with the centre of Taylor's Circle.

(Ibid, p. 83.)

(6.) Q, Q^1 and P are collinear.

(Aley—Contributions to the Geometry of the Triangle, p. 8.)

(7.) K, P^1 and Q_1 are collinear.

(Ibid, p. 13.)

(8.) S^1, K and D are collinear.

(Ibid, p. 15.)

(9.) H, M^1 and D are collinear.

(Ibid, p. 19.)

(10.) Z^1, F and D are collinear.

(Ibid, p. 24.)

(11.) Ω, Ω^1 and S are collinear.

(Schwatt—Geometric Treatment of Curves, p 7.)

(12.) K, Z, M are all collinear.

(Ibid, p. 3.)

(13.) Z^1, H, and S are collinear.

(Ibid, p. 13.)

(14.) N, M, and D are collinear.

(Ibid, p. 17.)

(15.) D, S and G are collinear.

(Ibid, p. 7.)

(16.) Q, O and G are collinear.

(Ibid, p. 36.)

(17.) D^1, H and N are collinear.

(Ibid, p. 16.)

(18.) Q, M and Z are collinear.

(Ibid, p. 44.)

(19.) R, O and M_1 are collinear.

(Ibid, p. 43.)

(20.) M_c, M_{1c} and S are collinear.

(Casey—Sequel, 5th edition, p. 242.)

(21.) K, M and the center of the triplicate ratio circle are collinear.

(Richardson and Ramsey—Modern Plane Geometry, p. 41.)

(22.) N, M and the point of concurrence of lines through A, B, C parallel to the corresponding sides of Brocard's first triangle are collinear.

(Lachlan—Modern Pure Geometry, p. 81.)

(23.) K, M_a and the middle point of altitude upon B C are collinear.

(Richardson and Ramsey—Modern Plane Geometry, p. 58.)

(24.) H, G and F are collinear.

(W. B. Smith—Modern Synthetic Geometry, p. 141.)

(25.) If A^1 is the pole of B C with respect to the circumcircle of A B C, then A_1, A and the Symmedian point are collinear.

(Casey—Sequel, 5th edition, p. 171.)

(26.) The intersections of the anti-parallel chords $D^1 E$, $E^1 F$, $F^1 D$ with Lemoine's parallels $D E^1$, $E F^1$, $F D^1$ respectively, are collinear. The D, E, F, D^1, E^1, F^1, are the six points of intersection of Lemoine's circle with the sides of the triangle.

(Ibid, p. 182.)

(27.) If the line joining two corresponding points of directly similar figures F_1, F_2, F_3 described on the sides of the triangle AB C, pass through the centroid, the three corresponding points are collinear.

(Ibid, p. 237.)

(28.) If from Tarry's point ⊥'s be drawn to the sides B C, C A, A B of the triangle, meeting the sides in $(\alpha, \alpha_1, \alpha_2)$ $(\beta, \beta_1, \beta_2)$ $(\gamma, \gamma_1, \gamma_2)$, the points α, β, γ are collinear, so also $(\alpha_1, \beta_2, \gamma)$ and $(\alpha_2, \beta, \gamma_1)$. (Neuberg.)

(Ibid, p. 241.)

(29.) In any triangle AB C, O, O^1 are the centres of the inscribed circle and of the escribed circle opposite A; O O^1 meets B C in D. Any straight line through D meets AB, AC respectively in b, c. O b, O^1 c intersect in P, O^1b, O c in Q. P A Q is a straight line perpendicular to O O^1.

(Wolstenholme—Math. Problems, p. 8, No. 79.)

(30.) A triangle P Q R circumscribes a circle. A second triangle A B C is formed by taking points on the sides of this triangle such that A P, B Q, C R are concurrent. From the points A, B, C tangents A a, B b, C c are drawn to the circle. These tangents produced intersect the sides B C, C A, AB, in the three points a b c, which are collinear.

(Catalan Geométrie Elementáire, p. 250.)

(31.) The three internal and three external bisectors of the angles of a triangle meet the opposite sides in six points which lie three by three in four straight lines.

(Richardson and Ramsey—Modern Plane Geometry, p. 19.)

(32.) If O be any point, then the external bisectors of the angles B O C, C O A, A O B meet the sides B C, C A, A B respectively in three collinear points.

(Ibid, p. 52.)

(33.) The external bisectors of the angles of a triangle meet the opposite sides in collinear points. (A special case of 31.)

(Lachlan—Modern Pure Geometry, p. 57.)

(34.) Lines drawn through any point O perpendicular to the lines O A, O B, O C meet the sides of the triangle AB C in three collinear points.

(Ibid, p. 59.)

(35.) If any line cuts the sides of a triangle in X, Y, Z; the isogonal conjugates of A X, B Y, C Z respectively will meet the opposite sides in collinear points.

(Ibid, p. 59.)

(36.) If a line cut the sides in X, Y, Z; the isotomic points of X, Y, Z with respect to the sides will be collinear.

(Ibid, p. 59.)

(37.) If from any point P on the circumcircle of the triangle AB C, P L, P M, P N be drawn perpendicular to P A, P B, P C, meeting B C, C A, AB, in L, M, N. then these points L, M, N are collinear with circumcentre.

(Ibid, p. 67.)

(38.) If P L, P M, P N be _'s drawn from a point P on the circumcircle to the sides B C, C A, AB respectively, and if Pl, Pm, Pn be drawn meeting the sides in l, m, n and making the angles L Pl, M Pm, N Pn equal when measured in the same sense, then the points l, m, n are collinear.

(Ibid, p. 68.)

(39.) If X Y Z and X^1 Y^1 Z^1 are any two transversals of the triangle AB C: Y Z′; Z X^1, X Y^1 cut the sides B C, C A, AB in collinear points.

(Ibid, p. 60.)

(40.) If X Y Z and X^1 Y^1 Z^1 be any two transversals of the triangle AB C, and and if Y Z^1, Y^1 Z meet in P, Z X^1, Z^1 X meet in Q, X Y^1, X^1 Y in R, then A P, B Q, C R cut the sides B C, C A, AB in collinear points.

(Ibid, p. 61.)

(41.) If the lines AO, BO, CO cut the sides of the triangle AB C in X, Y, Z; and if the points X^1, Y^1, Z^1 be the harmonic conjugate points of X, Y, Z with respect to B, C; C, A; and A, B, respectively, then X^1, Y^1, Z^1 are collinear.

(Ibid, p. 61.)

(42.) If the inscribed circle touch the sides in X, Y, Z, then the lines Y Z, Z X, X Y cut the sides B C, C A, AB in three collinear points.

(Ibid, p. 62.)

(43.) The feet of perpendiculars from H and G upon A G and A H respectively are collinear with K.

(Ibid, p. 147.)

(44.) If three triangles AB C, A_1 B_1 C_1, and A_2 B_2 C_2 have a common axis of perspective, their centres of perspective when taken two and two, are collinear.

(McClellan—Geometry of the Circle, p. 122.)

(45.) AB C is a triangle inscribed in and in perspective with A^1 B^1 C^1; the tangents from A B C to the incircle of A^1 B^1 C^1 meet the opposite sides in three collinear points, X, Y, Z (B C in X, etc.).

(Ibid, p. 128.)

(46.) If three pairs of tangents drawn from the vertices of a triangle to any circle, meet the opposite sides X, X^1, Y, Y^1, Z, Z^1, and if X, Y, Z are collinear, so also are X^1, Y^1, Z^1.

(Ibid, p. 128.)

(47.) If X Y Z is a transversal and if X^1, Y^1 Z^1 are the harmonic conjugates of X, Y, Z, then

Y^1, Z^1, X; Z^1 X^1, Y; X^1, Y^1, Z are collinear.

Also the middle points of X X^1, Y Y^1, Z Z^1 are collinear.

(Ibid, p. 131.)

(48.) If L is an axis of symmetry to the congruent triangles A B C and A^1 B^1 C^1 and O is any point on L, A^1 O, B^1 O, C^1 O cut the sides B C, C A, AB in three collinear points.

(Depuis—Modern Synthetic Geometry, p. 204.)

(49.) Two triangles which have their vertices connecting concurrently, have their corresponding sides intersecting collinearly.

(Desargue's Theorem.) (Ibid, p. 204.)

(50.) A^1, $B,^1$ C^1 are points on sides of AB C such that AA^1, BB^1, C C^1 are concurrent, then AB, A^1 B^1; B C, B^1 C^1, C A, C^1 A^1 meet in three points Z, X, Y which are collinear.

(Ibid, p. 205.)

(51.) If P be any point, AB C a triangle and A^1 B^1 C^1 its polar reciprocal with respect to a polar centre O, the perpendiculars from O on the joins P A, P B, and P C intersect the sides of A^1 B^1 C^1 collinearly.

(Ibid, p. 223.)

(52.) If the three vertices of a triangle be reflected with respect to any line, the three lines connecting the reflexions with any point on the line intersect collinearly with the opposite sides.

(Townsend—Modern Geometry, p. 180.)

(53.) When three of the six tangents to a circle from three vertices of a triangle intersect collinearly with the opposite sides, the remaining three also intersect collinearly with the opposite sides.

(Ibid, p. 180.)

(54.) If from the middle points of the sides of the triangle A B C, tangents be drawn to the corresponding Neuberg circles, the points of contact lie on two right lines through the centroid of A B C.

(Casey—Sequel, p. 241.)

(55.) If P is a Simson's point for A B C, and O any other point on the circumcircle of A B C, then the projections of O upon the Simson's lines of O with respect to the triangles P A C, P B C, P C A, A B C are collinear.

(Lachlan—Modern Pure Geometry, p. 69.)

(56.) When three lines through the vertices of a triangle are concurrent, the six bisectors of the three angles they determine intersect the corresponding sides of the triangle at six points, every three of which on different sides are collinear if an odd number is external.

(Ibid, p. 181.)

(57.) When three points on the sides of a triangle are collinear, the six bisections of the three segments they determine connect with the corresponding vertices of the triangle by six lines, every three of which through different vertices are collinearly intersectant with the opposite sides if an odd number is external.

(Ibid, p. 182.)

(58.) A_1, M_{1a} and K^1, the intersection of the Symmedian through A and the tangent to circumcircle at C, are collinear.

(Schwatt—Geometric Treatment of Curves, p. 4.)

(59.) If M X and F Y are parallel radii, in the same direction, in circumcircle and Feuerbach circle, then X, Y, and H are collinear.

(Ibid, p. 21.)

(60.) If Y_1 is the other extremity of the diameter F Y, then Y_1, G, and X are collinear.

(Ibid, p. 21.)

(61.) If P, a point on the circumcircle of A B C be joined with H^1, H^{11}, H^{111}, the respective intersections of the produced altitudes with circumcircle, and if the points of intersection of P H^1, P H^{11}, P H^{111} with B C, C A, A B be U, V, W respectively, then U, V, W are collinear.

(Ibid, p. 23.)

(62.) A O, B O, C O meet the circumcircle in A^1, B^1, C^1; perpendiculars from M upon the sides B C, C A, A B meet Nagel's circle in A^{11}, B^{11}, C^{11}; the corresponding sides of A^1 B^1 C^1 and A^{11} B^{11} C^{11} meet in three collinear points.

(Ibid, p. 40.)

(63.) The feet of the perpendiculars on the sides of a triangle from any point in the circumference of the circumcircle are collinear. (Simson's line.)

(64.) If two triangles are in perspective the intersections of the corresponding sides are collinear. A different statement of 49.

(Mulcahy—Modern Geometry, p. 23.)

(65.) The perpendiculars to the bisectors of the angles of a triangle at their middle points meet the sides opposite those angles in three points which are collinear.

(G. DeLong Champs.) (Mackay, Euclid, p. 356.)

I, I_1, I_2, I_3 are the centres of the inscribed and three escribed circles of the triangle A B C. D, E, F; D_1, E_1, F_1; D_2, E_2, F_2; D_3; E_3; F_3: are the feet of the perpendiculars from these centres upon the respective sides. N, P, Q are the feet of the bisectors of the angles A, B, C.

(66.) A B, D E, D_2 E_1 concur at Q_1.
B C, E F, E_3 F_2 concur at N_1.
C A, F D, F_1 D_3 concur at P_1.
Q_1, N_1, and P_1 are collinear.

(67.) A B, D_1 E_2, D_3 E_3 concur at Q_2.
B C, E_2 F_3, E_1 F_1 concur at N_2.
C A, F_3 D_1, F_2 D_2 concur at P_2.
Q_2, N_2, and P_2 are collinear.

(68.) A B, N P, I_1 I_2 concur at Q_3.
B C, P Q, I_2 I_3 concur at N_3.
C A, Q N, I_3 I_1 concur at P_3.
Q_3 N_3 and P_3 are collinear.

(66, 67, 68—Stephen Watson in Lady's and Gentleman's Diary for 1867, p, 72. Mackay, Euclid, p. 357.)

(69.) M_a, the middle point of Q O and the middle point of Q A are collinear.

(Mackay, Euclid, p. 363.)

(70.) The six lines joining two and two the centres of the four circles touching the sides of the triangle A B C, pass each through a vertex of the triangle.

(Mackay, Euclid, p. 252.)

(71.) M_a, O, and the middle of the line drawn from the vertex to the point of inscribed contact on the base are collinear. A similar property holds for the escribed centres.

(Mackay, Euclid, p. 360.)

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