To find the basis of the forms of $P_{\alpha, \beta, \gamma}$ we need the recursion formulæ, which may be verified by computation.

$$P_{\alpha, \beta, \gamma} = P_{\beta, \gamma, \alpha} = P_{\gamma, \alpha, \beta} \quad \ldots\ldots (7)$$

$$\Sigma Z_i^6 P_{\alpha, \beta, \gamma} - \Sigma Z_i^6 Z_k^6 P_{\alpha - 6, \beta, \gamma} + \Sigma Z_i^6 Z_k^6 Z_l^6 P_{\alpha - 12, \beta, \gamma} - \Pi Z_i^6 P_{\alpha - 18, \beta, \gamma} = P_{\alpha + 6, \beta, \gamma} \quad \ldots (8)$$

$$\Sigma Z_i^6 P_{\alpha, \beta, \gamma} - \Sigma Z_i^6 Z_k^6 P_{\alpha, \beta - 6, \gamma} + \Sigma Z_i^6 Z_k^6 Z_l^6 P_{\alpha, \beta - 12, \gamma} - \Pi Z_i^6 P_{\alpha, \beta - 18, \gamma} = P_{\alpha, \beta + 6, \gamma} \quad \ldots (9)$$

$$\Sigma Z_i^6 P_{\alpha, \beta, \gamma} - \Sigma Z_i^6 Z_k^6 P_{\alpha, \beta, \gamma - 6} + \Sigma Z_i^6 Z_k^6 Z_l^6 P_{\alpha, \beta, \gamma - 12} - \Pi Z_i^6 P_{\alpha, \beta, \gamma - 18} = P_{\alpha, \beta, \gamma + 6} \quad \ldots (10)$$

$$\Pi Z_i^6 \cdot \Sigma Z_i^6 Z_k^6 P_{\alpha, \beta, \gamma} - \Pi X_i^{12} \Sigma X_i^6 P_{\alpha - 6, \beta - 6, \gamma - 6} - \Pi Z_i^{18} P_{\alpha - 12, \beta - 12, \gamma - 12} + \Sigma Z_i^6 Z_k^6 Z_l^6 P_{\alpha + 6, \beta + 6, \gamma + 6} - P_{\alpha + 12, \beta + 12, \gamma + 12}.$$

Making $\gamma = o$ in 8, 9, 10, we get recursion formulæ for $P_{\alpha, \beta}$.

By a repeated and successive application of the formulæ 8, 9, 10 to any $P_{\alpha, \beta, \gamma}$, it is expressed as a rational integral function of forms $P_{\alpha', \beta', \gamma'}$ whose greatest index is 18, and therefore finite in number, and which is therefore the basis of the system.

I will add, however, that by a somewhat tedious reduction it can be shown that the system can be expressed as rational functions of

$$\Sigma Z_i^6, \Sigma Z_i^6 Z_k^6, \Sigma Z_i^6 Z_k^6 Z_l^6, Z_1 Z_2 Z_3 Z_4, P_{9,3}, P_{15,3}, P_{6,3,3}, P_{6,9,3}.$$

Rate of Decrease of the Intensity of Sounds With Time of Propogation. By A. Wilmer Duff.

[Abstract.]

Part I.—Theoretical.

The intensity of sounds spreading in spherical waves from a source would, if no part of the energy of vibration were lost in the passage, vary inversely as the square of the distance. But it is certain that a considerable proportion of the sound energy must in every second be converted into heat, though no attempt seems to have been made to determine experimentally what proportion this is of the whole. The transformation of energy of vibration into heat energy takes place in three ways. In the first place, the viscosity or internal friction of the air must cause a diminution of the vibrational energy and the production of heat. Again, in each condensed part of a sound wave heat is produced by the condensation; this causes a rise of temperature and an immediate tendency for radiation of heat and conduction of heat to take place. Similarly a fall of temperature takes place in the rarefied part of the wave with a similar tendency to radiation and conduction. Now, the greater the extent of this radiation and conduction, the less will be the amount of vibrational energy handed on and consequently the less the intensity of the sound. In addition to spherical spreading, viscosity, conduction and radiation, two other sources of diminution of intensity might be mentioned, namely, atmospheric refraction and lack of atmospheric homogeneity; but these two latter influences are only occasional or local, while the former are invariable and universal.

Several eminent physicists have given theoretical discussions of the effects of viscosity, radiation or conduction separately or of two of them simultaneously. Thus Stokes, in 1845, studied the effect of viscosity and deduced numerical results, and in 1851 he found a formula for the effect of radiation, but calculation from this result is still impossible because of our total ignorance of the rate of radiation of a gas. Rayleigh has applied Stokes' method to estimate the effect of conduction. These investigations referred to plain waves only. Kirchoff, in 1868, discussed the effect of conduction and viscosity together on plane waves, and indicated the result for spherical waves also. Brunhes has recently examined the effect of conduction on plane waves, obtaining a result in accord with those of Kirchoff and Rayleigh.

In order to deduce any intelligible results from the observations, I have made it necessary to either assume or establish a law of diminution of intensity in spherical waves, taking account of viscosity, radiation, conduction and spherical spreading simultaneously. As considerable doubt might attach to a general formula framed by the superposing the formulæ already obtained for the separate effects enumerated, it seemed advisable to undertake a new theoretical investigation for spherical waves affected by viscosity, radiation and conduction. (This discussion is here omitted, but will be printed in full elsewhere.) The conclusion which we arrive at is that the intensity varies as

$$\frac{e^{-2mr}}{r^2}$$

provided the proportional rate of fall of temperature of the gas by radiation be small compared with the number expressing the pitch of the sound. In the experimental work described in the second part of this paper sounds of very high pitch were employed, so that the above condition was satisfied. By this means a value for the constant radiation is finally deducted, and it is found to be small in comparison with the pitch-number of any ordinary sound, and so the solution obtained above may be considered as holding true, at least very approximately, for any sound of moderate pitch.

PART II.—EXPERIMENTAL.

Any attempt to compare sound intensities is attended by great difficulties. The term intensity can itself be understood in two ways—firstly in the subjective or physiological sense of loudness, and secondly, in the objective or dynamical sense of rate of flow of energy. The only case in which there seems to be a constant proportion between these two is when we compare sounds of the same pitch and quality. In deducing results from the experiments, I have made the following assumption: When two faint and diminishing sounds are indistinguishable by the ear as regards pitch and quality, the minimum of energy flow required for audibility is the same. While this assumption cannot be fully justified, it seems at least inherently highly probable, and can at most differ but slightly from the truth. The errors consequent on this assumption probably lie well within the experimental errors in the following method:

A large number of very small whistles were made of as nearly as possible the same shape and dimensions. From these the eight that seemed most similar were chosen and mounted on a wind-chest in such a way that any number could be blown under a definite pressure measured by a water monometer. The distances at which each pair and the whole eight just became inaudible were then determined. It was found that near the limits of audibility the sounds were indistinguishable in quality. Let R be the distance at which all eight whistles blown simultaneously became inaudible and r the mean distance at which two became inaudible. Then at a distance r the mean rate of energy-flow from two whistles equals the minimum for audibility, and hence that of eight whistles equals four times the minimum for audibility, while at distance R, the rate of energy-flow of eight whistles equals the minimum for audibility. If now, in accordance with the above assumption, the minimum be the same in both cases,

$$\frac{e^{-2mR}}{R^2} : \frac{e^{-2mr}}{r^2} :: 1 : 4$$

$$\text{and } \therefore m = \frac{\log_e \left(2\frac{r}{R}\right)}{R - r}$$

The observations were made at a very quiet place on the River St. John, in New Brunswick, Canada, the whistles being sounded on one side by an assistant and the sounds listened to on the opposite side. Only times when there was no appreciable wind were chosen for observation. When a wind sprang up in the course of a set of observations the work was discontinued and the observations discarded. In order to eliminate the effect of reflection from the banks, several different stations for sounding and directions for observing were tried in different sets of observations. The orifices of the whistles were always kept turned directly toward the observer. To avoid errors due to the tiring of the ear, in some cases the observations on the pairs of whistles were made first, and in other cases those on the eight whistles. Various devices were tried to eliminate the effect of bias in determining the distance of inaudibility. As regards the difficulty of determining the point at which the sounds became inaudible, this was found much less than was expected. Nevertheless, there was always a space in which it was doubtful whether the sound was heard or imagined. The middle point of such a space was taken as the most probable position of extinction.

As regards the success attained in attempting to make the different pairs of whistles of equal intensity, the following figures may be quoted. (These were the only cases in which the position for each pair of whistles was finally calculated. In general the position for each pair was marked by a stake and the mean finally taken for measurement.) In the first the mean distance was 1,693 feet and the distance of the separate observationstions for the mean —28, —18, +5, +41, a total range of 69 feet. In the other the mean distance was 2,078 feet, and the distances from the mean were —39, —15, +9, +45, a total range of 84 feet. Here something must be allowed for unavoidable bias, but yet the result seems fairly satisfactory.

The whistles were made of the stopped-organ-pipe form. In the absence of facilities for the purpose their exact pitch was not determined until after the experiments had been made. The pitch was then determined by using a high pressure sensitive flame to find the nodes of the stationary waves produced by reflection from a wall. The semi-wave lengths are as follows, in inches: .98, .97, .95, .95. .97. .95. .95. .96. The mean of these is .960, corresponding to a vibration frequency of 6.820. The following table gives the series of readings made and the values of m deduced, all lengths being in feet:

Series.	r.	R.	Temp.	m.
1	1260	1660	80° F	.0010
2	1333	1699	74	.0012
3	2078	2473	60	.0013
4	1502	1834	70	.0015
5	1959	2335	68	.0013

To enable us to appreciate the meaning of these figures, it may be noted that, since the eight whistles gave a sound of four times as great an intrinsic intensity as the mean pair of whistles, this sound should, if no cause except spherical spreading affected the intensity, be audible just twice as far, while, as a matter of fact, owing to the other causes enumerated, it was usually audible only about one and one-quarter times as far. The value of m may be expressed by saying that, excluding the effect of spherical spreading, the intensity of the sound died off about one-fourth of 1 per cent. for each foot of advance.

For such rough determinations of a quantity very difficult to determine at all, the agreement between the values of m, as shown by the last column, seems very satisfactory.

If now we return to the theoretical investigation it can be shown that the value of m consists of the sum of three parts, due, respectively, to viscosity, conduction and radiation. Since the constants of viscosity and conduction are known, while the full value of m has been found by expeiment, it can be calculated that the effect of viscosity is to produce a diminution of intensity amounting to one-fortieth of 1 per cent. per foot of advance; of conduction, one-seventieth of 1 per cent., while the remainder, amounting to one-fifth of 1 per cent., is due to radiation.

It may be questioned whether all of the part of the effect thus attributed to radiation is really due to that cause and whether part of it may not be due to refraction and heterogeneity of atmosphere, as stated earlier. That it is not due to refraction is pretty certain, for refraction tends to produce actual sound shadows of the surface of the earth or water at certain distances, and so would be very marked when acting. It also varies greatly with the gradient of temperature, being nonexistant when the temperature does not change as we ascend for some distance. Now, both the distances of observation and the state of the atmosphere would vary greatly, and hence if the circumstances were such that refraction had any considerable effect the values obtained for m would vary widely among themselves. Moreover, calculation will show that for the elevation of sounding station and observing station employed, the distances at which refraction would appreciably affect the value of m would be much greater than those employed in the observations.

As regards invisible striaè of water vapor, these must have had a very small effect, if any at all, for the effect should vary very greatly with the state of the atmosphere, and this varied very widely, some observations being made at noonday of very hot days, others between 7:30 and 8:30 in the evening, and on one occasion, while the sky was overcast and the atmosphere heavily charged with water vapor just before a storm. Probably, therefore, nearly all the effect mentioned is due to radiation.

Now, the theoretical investigation shows that the effects of viscosity and conduction decrease very rapidly with decreasing pitch, being proportional to the squares of the vibration frequency, while the effect of radiation is independent of pitch. We conclude, therefore, that for sounds of ordinary pitch the effect of viscosity and conduction on intensity is practically negligible, while the effect of radiation amounts to about one-fifth of 1 per cent. per foot of advance.

The Constant of Radiation of Air. By A. Wilmer Duff.

[Abstract.]

Assuming that for small differences of temperature, radiation takes place according to the law stated by Newton, namely, in proportion to the excess of temperature of the radiating body above its surroundings, we may define the constant of radiation of air as the rate of cooling (by radiation) of a body of air which has been raised one degree above the sounding mass. From the results of the preceding paper it can be calculated that this is about 8.3 degrees per sec. This is equivalent to saying that in the first hundredth of a second the heated mass would

cool one-twelfth of a degree, or if a mass of air be heated to any excess above the surrounding mass it will fall to one-half of that excess in about one-twelfth of a second.

For reasons stated in the preceding paper it is evident that this value must err rather in being too large than in being too small.

Preliminary Results by a New Method for the Study of Impact.*

By A. W. Duff and J. B. Meyer.

The purpose of this paper is to briefly describe an apparatus for the study of impact of masses of wood on one another, and to state a few results obtained by means of it. It was intended to include in the investigation not only the change in the relative velocity of the impinging bodies produced by impact, but also the length of time the bodies are in contact, the closeness of approach produced by their mutual compression, and the internal vibrations to which impact gives rise. The apparatus was constructed by Mr. Meyer, and the present results obtained by him early in this year, but only a small part of the contemplated work was completed when it had to be discontinued. Calling, as usual, the ratio of the velocity of separation after impact to the velocity of approach before impact the coefficient of restitution, it may be stated that the results to be given here are only a few isolated determinations of the coefficient of restitution and of the time of contact.

In principle the apparatus consists of a block dropped vertically on a much larger mass of the same material, the circumstances of the impact being recorded in a curve traced by a pencil attached to the block on a vertical revolving drum covered with a sheet of paper. To describe the apparatus more fully, it consists of two vertical beams mortised in a massive cross-shaped base, the beams adjustable so that the space between can be regulated. The height of the beams is 8 feet. Between these beams as guides the block can descend with comparatively little friction. On the base, the larger of the two impinging masses, or the plate as we shall call it, is rigidly clamped. Immediately in front of the beams is fixed on cone bearings a vertical rotating cylinder around which a sheet of paper is wrapped. The cylinder is 2 feet in diameter and 2½ feet in height. Fastened to the top of the descending block is a small removable board, to which is attached a brass tube carrying a pencil. The tube is secured in position by a catch attached

*This paper is an abstract of a thesis presented by Mr. J. B. Meyer for the degree of B. Sc., and placed in the library of Purdue University.