by the lake. The international exposition of 1893 will epitomize in material form the progress of the world for the centuries, and to no Mecca can the devotee of science turn with more reverent steps.

The interdependence of the liberal pursuits there will have practical illustrations of the most instructive character. The best thought of the centuries will be realized on canvass, in marble, in bronze, in exquisite fabrics, in jewels and ornaments of silver and gold, in the whirr of machinery and the flashes of electricity.

There may we study things, and there may we in profitable intercourse meet men. This will be the academy of science of the world.

PAPERS READ.

On the construction of a sensitive galvanometer. By Benj. W. Snow.

Tests of the torsional strength of a steel shaft. By Thos. Gray.

Analytical and quaternion treatments of the problem of sun and planet. By A. S. Hathaway.

INTRODUCTION.

The object of the paper is to show the greater simplicity of quaternions over analytics. For the purpose of comparison, the most condensed analytical treatment possible is adopted. This turns out to be precisely analagous to the quaternion treatment. Three equations, such as $m\,a=a'$, $m\,b=b'$, $m\,c=c'$ are written $m\,(a, b, c)=(a', b', c')$. By multiplying these equations by (x, y, z) is understood the result of multiplying the first by x, the second by y, the third by z, and adding, giving $m\,(a\,x+b\,y+c\,z)=(a'\,x+b'\,y+c'\,z)$. This corresponds to scalar multiplication in quaternions. By forming corresponding determinants with x, y, z, is understood the set of equations $m \begin{matrix} a\,b\,c \\ x\,y\,z \end{matrix} = \begin{matrix} a'\,b'\,c' \\ x\;y\;z \end{matrix}$ or, in full $m\,(b\,z-c\,y,\ c\,x-a\,z,\ a\,y-b\,x)=(b'\,z-c'\,y,\ c'\,x-a'\,z,\ a'\,y-b'\,x)$. This corresponds to vector multiplication in quaternions.

The analytical methods thus perfected are, in fact, a sort of degraded and cumbersome quaternion notation in which (a, b, c) stand for $ai+bj+ck$, etc. It involves the necessity of thinking by steps parallel to the axes, and when results are obtained it involves the fitting together of the various steps in order to see what is the actual state of affairs in space. To do this requires considerable practice and grasp of technique, all of which is avoided in quaternions. For example, equations (8) were unnecessary in quaternions, the results desired being sufficiently evident from (7); while even after (8) is derived the technique of equations of the first degree must be at command before the results stated can be seen in the analytical method. The letters m_1, m_2 in (9) and on are not the masses of (1) . . . (5).

EQUATIONS OF MOTION.

$$(1)\quad m_1\left\{\frac{d^2x_1}{dt^2},\frac{d^2y_1}{dt^2},\frac{d^2z_1}{dt^2}\right\}=\frac{m_1m_2}{r^3}(x,y,z) \qquad m_1\frac{d^2\rho_1}{dt^2}=\frac{m_1m_2}{T^3\rho}\rho$$

$$(2)\quad m_2\left\{\frac{d^2x_2}{dt^2},\frac{d^2y_2}{dt^2},\frac{d^2z_2}{dt^2}\right\}=\frac{m_1m_2}{-r^3}(x,y,z) \qquad m_2\frac{d^2\rho_2}{dt^2}=-\frac{m_1m_2}{T^3\rho}\rho$$

where $(x,\ y,\ z)=(x_2-x_1,\ y_2-y_1,\ z_2-z_1)$ where $\rho=\rho_2-\rho_1$
and $r=(x^2+y^2+z^2)^{\frac{1}{2}}$

Adding (1), (2), also dividing out common m's and subtracting, putting $M=m_1+m_2$, we have:

$$(3)\quad \left(m_1\frac{d^2x_1}{dt^2}+m_2\frac{d^2x_2}{dt^2},\ldots\ldots\right)=(0,0,0) \qquad m_1\frac{d^2\rho_1}{dt^2}+m_2\frac{d^2\rho_2}{dt^2}=0$$

$$(4)\quad \left\{\frac{d^2x}{dt^2},\frac{d^2y}{dt^2},\frac{d^2z}{dt^2}\right\}=-\frac{M}{r^3}(x,y,z) \qquad \frac{d^2\rho}{dt^2}=-\frac{M}{T^3\rho}\rho$$

EQUATIONS OF MOTION INTEGRATED.

Integrating (3) twice, we have:

$$(5)\quad (m_1x_1+m_2x_2,\ \ldots,\ \ldots)=(at+b,\ a't+b',\ a''t+b'') \qquad m_1\rho_1+m_2\rho_2=\alpha t+\beta.$$

Hence, the center of gravity moves in a straight line with uniform speed, viz:

In the direction $a:a':a''$ with speed: $\sqrt{(a^2+a'^2+a''^2)}\cdot(m_1+m_2)$.

In the direction α, with speed: $T\alpha : M$.

Form corresponding products of (4) and $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ add and integrate.

$$(6)\quad \frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right] = \frac{M}{r} - \frac{M}{2a}$$

Multiply (4) by ρ and scalar-integrate:

$$\frac{1}{2}T^2\frac{d\rho}{dt} = \frac{M}{T\rho} - \frac{M}{2a}$$

This is the equation of energy. It shows that the speed of a planet increases when its distance from the sun decreases, and *vice versa*. Also, since $M = m_1 + m_2$ is sensibly the same for all planets, therefore the speed of a planet depends only on its distance from the sun and a constant, $2a$, of its orbit (later shown to be its major axis).

Forming corresponding determinants of (4) with (x, y, z) and integrating:

$$(7)\quad \begin{vmatrix} x & y & z \\ \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt}, \end{vmatrix} = c\,(l, l_1, l_2.)$$

where $l^2+l_1^2+l_2^2=1$ and c is positive.

Multiplying (4) by ρ and integrating the vector part:

$$V\rho\frac{d\rho}{dt} = c\lambda$$

where $Tc\lambda = c$.

Multiplying corresponding terms by (x, y, z), and adding, we find:

$$(8)\quad \begin{cases} l\,x + l_1 y + l_2 z = 0;\ \text{similarly,} \\ l\frac{dx}{dt} + l_1\frac{dy}{dt} + l_2\frac{dz}{dt} = 0. \end{cases}$$

Taking the scalar product by ρ we find

$S\lambda\rho = 0$; similarly

$$S\lambda\frac{d\rho}{dt} = 0.$$

Equation (7) shows the rate of description of double areas by the radius vector from sun to planet to be constant ($=c$) and that its motion is in a plane perpendicular to $(l:l_1:l_2)=\lambda$. The direction of this axis is such that an ordinary screw, when made to advance along it, will rotate in the direction of the description of areas.

Taking the second member of (7) with the first member of (4) and *vice versa*, and forming corresponding determinants and integrating, we have

$$(9)\ c \left\| \begin{matrix} l & l_1 & l_2 \\ \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{matrix} \right\| = - \frac{M}{r}(x, y, z) - f(m, m_1, m_2)$$

where $m^2 + m_1^2 + m_2^2 = 1$ and f is positive.

Multiplying the second member of (7) into the first member of (4) and *vice versa* and integrating, we have:

$$c \lambda \frac{d\rho}{dt} = - \frac{M}{T\rho} \rho - f \mu$$

where $T f \mu = f$.

Multiplying (9) by (l, l_1, l_2) and adding, we have $l\, m + l_1\, m_1 + l_2\, m_2 = 0$, or (m, m_1, m_2) is in the plane of motion.

Taking the scalar product by λ, we find $S \lambda \mu = 0$, or μ is in the plane of motion.

Take $(n, n_1, n_2) = \left\| \begin{matrix} l & l_1 & l_2 \\ m & m_1 & m_2 \end{matrix} \right\|$ forming the direction cosines of a third axis perpendicular to the two already found.

Take $\nu = \lambda \mu$ forming the rectangular unit vectors λ, μ, ν.

Form with (l, l_1, l_2) and (9) corresponding determinants, and we have:

$$(10)\ c \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \frac{M}{r} \left\| \begin{matrix} l & l_1 & l_2 \\ x & y & z \end{matrix} \right\| + f(n, n_1, n_2)$$

Multiply (9) by λ and we have:

$$c \frac{d\rho}{dt} = \frac{M}{T\rho} \lambda \rho + f \nu$$

This is the hodograph. It is a circle [remembering (8)] of radius $\frac{M}{c}$ and center $\frac{f}{c}(n, n_1, n_2) = \frac{f}{c} \nu$. The radius of this hodograph is one right angle in advance of the radius vector of the planet to which it corresponds.

Transposing the f terms of (9) to the first member, squaring, and using (6), we have:

$$(11)\ \frac{c^2 M}{a} + f^2 = M^2 \text{ or } a = c^2 M \mid (M^2 - f^2).$$

Multiplying (9) into (x, y, z) we have, by adding:

$$(12)\ c^2 - M r = f(m\, x + m_1\, y + m_2\, z).$$

Multiplying (9) into ρ and taking scalars:

$$c^2 - M T \rho = - f S \mu \rho$$

This, remembering (8), is the equation of the orbit. It is a conic whose focus is the sun, and axis is $(m, m_1, m_2) = \mu$. The eccentricity is $e = \frac{f}{M}$, the semi-parameter, $p = \frac{c^2}{M}$. Hence, the semi-major axis is $c^2 M | (M^2 - f^2)$, or a by (11). The center is $-a e (m, m_1, m_2) = -a e \mu$. We may put the orbit, therefore, in the form:

$$\rho = -a e \mu + \mu a \cos E + \nu b \sin E. \qquad e < 1.$$

$$\rho = -a e \mu + \mu a \cosh E + \nu b \sinh E. \qquad e > 1.$$

This substituted in (7) and integrated gives Kepler's equation

$$(13) \quad \begin{aligned} E - e \sin E &= \frac{c}{a b}(t - t_0) \qquad e < 1. \\ E - e \sinh E &= \frac{c}{a b}(t - t_0) \qquad e > 1. \end{aligned}$$

For analytical treatment see Dr. Dzisbek's Theories of Planetary Motion, pp. 1–13.

NOTES CONCERNING TESTS OF THE PURDUE EXPERIMENTAL LOCOMOTIVE. By WM. F. M. GOSS.

The Purdue experimental Locomotive Plant was installed early in the present year. It has been fully described in a paper read before the American Society of Mechanical Engineers at its San Francisco meeting, and a brief reference to the plan of mounting must serve the present purpose.

The driving wheels of the locomotive rest upon other wheels which are carried by shafts running in fixed bearings. When, as in the process of running, the drivers turn, their supporting wheels are driven by rolling contact. The locomotive as a whole instead of moving forward, remains at rest while the track, that is, the periphery of the supporting wheels, moves rearward. The locomotive draw-bar is connected with a series of scale-beams which constitute a traction dynamometer. Friction brakes on the shafts of the supporting wheels, interpose a resistance to the turning of the latter and, by so doing, supply a load for the locomotive. The whole arrangement is such that while the locomotive is fired in the usual way, it may be run under any load and at any speed, the conditions being similar to those of the track.