right angles also passes through these points. Conceiving the step m. O A drawn from A′ we see that M A and A′ M, A″ M are the real and imaginary components of the roots. The roots given by A′ and A″ are by the figure $-\frac{1}{2} - 1.3m$ and $-\frac{1}{2} + 1.3m$.

(c.) Real root of $2x^3 + 4x^2 + 8 - 4 = o$.

We have $OA = 2$, $AC = -4m$, $BC = 8m^2 = -8$, $CD = -4m^3 = -4m$. The circuit O A′ B′ D was drawn by aid of transparent paper turned round a pin with cross section paper underneath, after the manner of Lill's wooden and ground glass discs. The root, A′ A : m O A = tan A′ O A, may be read off from the cross section paper to several decimal places. It is here — .64....

O A′ B′ D is the circuit for the quadratic equation that gives the remaining pair of roots of the cubic. The circle on O D as diameter will not cut A′ B′ so that these roots are imaginary.

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ON SOME THEOREMS OF INTEGRATIONS IN QUATERNIONS. By A. S. HATHAWAY.

There are certain identities among volume, surface and line integrals of a quaternion function $q = f(h)$ that include as special cases the well known theorems of Green and Stokes, that are so often employed in mathematical physics. These indentities were first demonstrated by Prof. Tait by the aid of the physical principles usually employed in forming the so-called "Equation of Continuity." [See Tait's Quatermous, third ed., ch. XII J.]

If dh $d_1h$, $d_2h$ be non-coplanar differentials of the vector h, the theorems may be written:

(1) $-\iiint S\,dh\,d_1h\,d_2h.\nabla q = \iint \mathbf{V}\,dh\,d_1h.q$

(The surface integral extends over the boundary of the volume integral and $\mathbf{V}dhd_1h$ is an outward facing element of the surface.)

(2) $\iint \mathbf{V}(\mathbf{V}dhd_1h.\nabla).q = \int dhq$

(The line integral extends over the boundary of the surface integral in the positive direction as given by the vector areas $\mathbf{V}\,dhd_1h$.)

These theorems are analogous to the elementary theorem,

(3) $\int_A^B dq = q_B - q_A$ or in quaternion notation,

$-\int S dh \nabla.q = q$

It has not been noticed, so far as I am aware that these identities are equivalent to simpler identities pertaining to the operator $\nabla$, as follows:

(1)′ $Sdhd_1hd_2h.\nabla = Vd_1hd_2hSdh\nabla + Vd_2hdhSd_1h\nabla + Vdhd_1hSd_2h\nabla$

(2)′ $V(Vdhd_1h.\nabla) = dhSd_1h\nabla - d_1hSdh\nabla$

In fact (1) and (2) become these (into q) when applied to the elements of volume and surface just as (3) becomes $Sdh\nabla = d_1$ (into q) when applied to the element of length.

To identify (1) and (1)′, let h be the vector of the mean point of the parallelopiped whose edges are $dh, d_1h, d_2h$. The outward vector areas of the two faces parallel to $d_1h, d_2h$ are $-Vd_1hd_2h, Vd_1hd_2h$, and the corresponding values of q are $q+\frac{1}{2}Sdh.\nabla.q$, $q-\frac{1}{2}Sdh\nabla.q$; so that sum of the vector areas into q is $-Vd_1hd_2hSdh\nabla.q$. Similarly for the other faces.

So to identify (2) and (2)′, the line elements bounding the parallelogram $dh, d_1h$ are $dh, d_1h, -dh, -d_1h$, and the corresponding values of q are $q+\frac{1}{2}Sd_1h\nabla.q$, $q+\frac{1}{2}Sdh\nabla.q$, $q-\frac{1}{2}Sd_1h\nabla.q$, $q-\frac{1}{2}Sdh\nabla.q$ and the sum dbq is $dhSd_1h\nabla.q - d_1hSdh\nabla.q$.

To obtain (1) from (1)′ divide the given volume into infinitesimal parallelopipeds by any three systems of surfaces, one of which includes the boundary of the volume. In summing the terms (1)′ the introduced interior surfaces between adjacent elements of volume are gone over twice with the vector areas oppositely directed. These surfaces balance one another, therefore, and may be dropped from the summation, leaving the volume integral equal to the surface integral over the boundary of the volume integral.

We see also that if any discontinuity in q or its derivatives exists within the given volume that the proper way to overcome this is to surround the discontinuity by surfaces and so exclude the discontinuity. Usually this alters only the surface over which the surface integral extends without affecting the volume integral.

Similarly (2) is obtained from summation of (2)′ and, as every student of integral calculus is aware, (3) is obtained from dq in a similar manner.

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THE SECTIONS OF THE ANCHOR RING. By W. V. BROWN.