LABORATORY AND FIELD WORK ON THE PHOSPHATE OF ALUMINA. By H. A. HUSTON.

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THE DIGESTIBILITY OF THE PENTOSE CARBOHYDRATES. By W. E. STONE.

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A GRAPHICAL SOLUTION FOR EQUATIONS OF HIGHER DEGREE, FOR BOTH REAL AND IMAGINARY ROOTS. By A. S. HATHAWAY.

*1. Preliminary on imaginary numbers.

The usual idea of imaginary numbers, as presented in our text books of algebra, is that they are symbols introduced for the sake of making the laws of algebra formally complete. It is implied in the name given to these numbers that they have no actual meaning. This is a mistake. The failure to mean anything in ordinary cases is not the fault of the numbers, but results from the nature of the concrete quantities with which they are generally used. Like difficulties are experienced with real numbers under similar circumstances. Let us go briefly over the list of numbers and emphasize this point.

First, the numbers 1, 2, 3, 4, that denote repetitions of a concrete quantity. If the quantity be incapable of the indicated repetition the result is imaginary. Thus: Three spaces of four dimensions. This may be comprehensible to a different order of beings, but not to us.

Second, the numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, that denote partitions of a concrete quantity. Nevertheless, a space of $\frac{1}{2}$ a dimension, a school of $\frac{1}{4}$ a student, are impossibilities.

Third, the number $-1$. This number must be used with quantities of two kinds such that two of equal magnitude and different kinds give, when

*NOTE.—This preliminary on the graphic representation of imaginary numbers was not presented to the Academy. It is a simple and direct presentation of the subject without the use of analytical geometry, and on that account may be interesting to mathematicians; at the same time, it places the whole article upon an elementary basis, and makes it available to a larger circle of readers.

combined, zero result; e. g., assets and liabilities. In this case —1 reverses quality without altering magnitude, so that $1 + (-1) = 0$. But what is a farm of —80 acres? Imagine a farm that put with an 80 acre farm gives no land at all.

Fourth, the incommensurable numbers, e. g., the ratio of a diagonal to a side of a square. These require continuous quantity, and their use with quantity whose partitions are limited is impossible. What is a space of $\sqrt{2}$ dimensions, a country with $\sqrt{2}$ presidents, a man with $\sqrt{2}$ dollars in his pockets?

We recognize a number by what it can do with appropriate quantity to operate upon, not by what it can not do with inappropriate quantity. The interpretation of imaginary number requires quantity that has magnitude and different qualities. These quantities, whether geometrical or physical, may be represented by certain geometrical quantities called by Clifford steps.

The step from a position A in space to another position B has length and direction. Two steps are equal that have the same length, and the same direction; i. e., the opposite sides of a parallelogram taken in the same direction are equal steps. The sum of any number of successive steps in various directions is the step from the first point of departure to the last point reached; e. g., $AB + BC + CD = AD$. In particular the sum of two successive steps along the sides of a parallelogram is equal to the step along the diagonal. As the remaining sides in the parallelogram form equal steps added in reverse order, we learn that the order of successive steps in a sum may be changed without altering the sum.

Positive numbers operating on steps change lengths but not directions; —1 reverses direction without altering length; e. g., $-1\,AB = BA$. If x be any real number we see by similar triangles that $x\,(AB + BC) = x\,AB + x\,BC$.

A valuable analysis may be developed by the use of steps and real numbers only. From its simplicity, and its value in physical applications, it ought to displace ordinary analytical geometry, in technical schools at least. The main difficulty is the lack of a suitable text book.

Let us confine ourselves, now, to steps in the plane of the paper, and consider the nature of the number that multiplying O A produces O B. It must alter the length of O A into the length of O B; this is the tensor factor, an ordinary positive number. It must turn O A thus lengthened into O B; this is the versor factor; the angle of this turn, reckoned as positive when it is counter clockwise, is the angle of the number. Thus, let (2, 30°) denote a number that doubles length and turns 30° counter clockwise. Its tensor is 2, its versor is (1, 30°), and its angle is 30°.

After multiplying a step by (2, 30°) multiply the result by (3, 20°). Plainly the final step is (6, 50°) times the first step. This example of a product enables us to see at once that:

The tensor of a product equals the product of the tensors of the factors; and the angle of a product equals the sum of the angles of the factors. Hence the factors may be combined in any order without altering their product.

The definition of a sum of two numbers p and q is that (p + q) O B = p O B + q O B.* Replacing O B by r O A we have that (p + q) r = p r + q r; and since the factors of a product have been shown to be interchangeable, therefore r (p + q) = (p + q) r = r p + r q.

We thus find that these versi-tensors follow the ordinary laws of algebraic combination. To identify them with imaginaries, notice that $(1, 90°)^2 = (1, 180°) = -1 = (1, -90°)^2$. These two square roots of —1 are negatives of each other, for —1 (1, —90°) = (1, 180°) (1, —90°) = (1, 90°). So —1 has three cube roots, —1 and (1, ± 60°); and so on.

It is convenient to represent versi-tensors by steps. Some step O A is taken to represent unity; and then any other step represents its ratio to the unit step O A. Thus, if O B, O $B^1$ are steps of the same length as O A, and make angles of 60° and —60° respectively with O A, they represent the imaginary cube roots of —1. We may use geometry to put these roots in the standard form x + y i, where x and y are real numbers and i = (1, 90°). Let B $B^1$ meet O A in C; then O C represents, or say =, $\frac{1}{2}$, and C B = $\frac{1}{2}\sqrt{3}\,i$ = — C $B^1$; and from O B = O C + C B, O $B^1$ = O C + C $B^1$, we have $(1, \pm 60°) = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$.

This example just given makes it plain that any imaginary number may be put in the form x + y i, in one and only one way; and from the right triangle involved, we also see that the tensor of x + y i is $\sqrt{x^2 + y^2}$, the so-called modulus in imaginaries. It is easy to show by geometry how it is that every equation with real or imaginary co-efficients has at least one root, and therefore just as many roots as its degree and no more, or even to show the whole directly.* In fact, all the fundamental properties of imag-

* To see that this does define the sum, try it for the case of p = (2, 30°), q = (2, 150°), which gives p + q = (2, 90°). Also compare with the verification that 2 + 3 = 5.

inaries may be made visible realities rather than symbolic results based upon certain assumptions.

When dealing with steps not limited to the plane of the paper, then (O A, $n^\circ$) may be taken as the symbol of a number that turns any step that is perpendicular to O A, $n^\circ$ round O A as axis, counter clockwise to an observer at A, and lengthens in the ratio of the length of O A to the unit length. This is a quaternion. Quaternions whose angles are o° or 180° are ordinary positive and negative numbers, and are called scalars. Quaternions whose angles are 90° are called vectors. The square of a vector is a negative scalar. The ordinary rules of algebra hold except that factors are not interchangeable without altering the product. A quaternion, also, cannot multiply a step that is not perpendicular to its axis. It can be geometrically represented only by two steps. A vector (O A, 90°) or briefly (O A) may be represented by the step O A. The value of this representation is expressed by the equations:

$$(OB) + (OA) = (OB + OA)$$
$$(OB) : (OA) = OB : OA.$$

The calculus of quaternions is superior for all purposes of investigation to analytical geometry, and as its results can be immediately turned into analytical formulas, it is likely to be very much used and developed in the future. It is especially valuable in mathematical physics. An account of the system by Sir Wm. Rowan Hamilton, the inventor, was first presented to the Royal Irish Academy in 1843. The first book upon the subject, "Hamilton's Lectures," appeared in 1853.

II.

Let $a\,x^3 + b\,x^2 + c\,x + d = o$ be an equation with general imaginary co-efficients. Divide this by $x - r$: the quotient is $a\,x^2 + (a\,r + b)\,x + (a\,r^2 + b\,r + c)$ and the remainder is $a\,r^3 + b\,r^2 + c\,r + d$. The co-efficients of the quotient, and final remainder are best found by synthetic division, which shows the general method of forming each co-efficient by multiplying the last by r and adding the next co-efficient of the original equation. The process is identical with the reduction of the compound number (a, b, c, d) whose radix is r. The test of a root is that the remainder should be zero.

The steps that represent these numbers may be constructed as follows:

Take in the plane of the paper steps O A, A B, B C, C D, representing the numbers a, b, c, d. Take any point A′, and let A′ A : O A be the r we are to try in the equation for x. To find the result of the trial, construct the triangle A′ B′ B similar to O A′ A, and then the triangle B′ C′ C, also similar to O A′ A. We have O A = a, A′ A = a r, and hence A′ B = A′ A + A B = a r + b; also by similar triangles, B′ B = r A′ B = a $r^2$ + b r, and hence B′ C = B′ B + B C = a $r^2$ + b r + c. Again by similar triangles, C′ C = r (a $r^2$ + b r + c) = a $r^3$ + b $r^2$ + c r and hence C′ D = C′ C + C D = a $r^3$ + b $r^2$ + c r + d, the remainder sought; moreover, the co-efficients of the quotient are represented by O A, A′ B, B′ C. The problem is to so choose the first point A′ that the last vertex C′ of the series of similar triangles O A′ A, A′ B′ B, B′ C′ C, shall coincide with D: then A′ A : O A is a root of the given equation. With the ability to construct a series of similar triangles with ease, a position for A′ may be approximated to without much difficulty. Observe that O A′, A′ B′, B′ C′ are equi-multiples of O A A′ B, B′ C. This follows from the similar triangles O A′ A, A′ B′ B, B′ C′ C, which give O A′ : O A = A′ B′ : A′ B = B′ C′ : B′ C both as to tensor and angle parts. Hence the circuit O A′ B′ C′ represents the quotient on the new scale in which O A′ instead of O A represents the first co-efficient a.

If the co-efficients of the given equation are all real numbers and only the real roots are sought, the method fails, since A′ must be taken on O A produced giving no triangle O A′ A. In such a case, put x = $\frac{z}{m}$ where m is a given versor, say (1, 60°), or (1, 90°); the equation becomes:

$$a z^3 + m b z^2 + m^2 c z + m^3 d = o.$$

The figure O, A, B, C, D that represents the co-efficients of this equation will have equal angles at A, B, C, viz.: the supplement of the angle of m (since a, b, c, d are real numbers, their angles are 0 or 180°). We are to seek for roots of this equation whose angles are, angle of m or angle of m + 180°. (Since z = m x, therefore angle z = angle m + angle x.) Thus A′ must be taken on A B produced; and since the angles at A, B, C, are equal, it follows that the similar triangles required will have their vertices B′, C′ on B C, C D, produced, so that the construction of these triangles is simplified, e. g., to find B′, draw from A′ a line making with O A′ an angle equal to the angle A; that line meets B C in B′. The broken line O A′ B′ C′ has its angles A′, B′ equal to the angles A, B, and its vertices A′, B′, C′ in the sides A B, B C, C D; trials of this construction must be made until C′ co-incides with D, when A′ A : m O A is the real root of the equation in x.

Taking m = (1, 90°), this is Lill's construction for the real roots of an equation with real co-efficients. Lill has devised an instrument for facilitating his construction, which is described as follows (Cremona Graph. Statics (Beare), p. 76):

"The apparatus consists of a perfectly plane circular disc, which may be made of wood; upon it is pasted a piece of paper ruled in squares. In the center of the disc, which should remain fixed, stands a pin, around which as a spindle another disc of ground glass of equal diameter can turn. Since the glass is transparent, we can with the help of the ruled paper underneath, immediately draw upon it the circuit corresponding to the given equation. If we now turn the glass plate, the ruled paper assists the eye in finding the circuit which determines a root. A division upon the circumference of the ruled disc enables us by means of the deviation of the first side of the first circuit from the first side of the second, to immediately determine the magnitude of the root. For this purpose the first side of the circuit corresponding to the equation must be directed to the zero point of the graduation."

Linkages might be found to perform mechanically what must be done by successive approximations in the method above, viz.: to bring the last vertex C′ into co-incidence with D. Kempe has given some linkages for a different construction. [See Messenger of Mathematics, Vol. 4, 1875, p. 124.]

## III.

The following constructions are given as illustrations:

(a.) Roots of $2x^2 + 4x + 1 = 0$. [Fig. I.]

As the co-efficients are all real it is preferable, and for real roots necessary, to transform the equation by putting $x = \frac{z}{m}$, $m = (1, 90°)$. The equation becomes $2z^2 + 4mz + m^2 = 0$, and $OA = 2$, $AB = 4m$, $BC = m^2 = -1$. If A′ A : O A is a root of this equation then, dividing by m, we find A′ A : m O A as a root of the original equation. If this is real A′ must lie on A B, produced if necessary. Remember that A′ is such that O A′ A, A′ C B are similar triangles and we see that the angle O A′ C is a right angle when A′ lies on A B. Hence the circle on O C as diameter cuts A B in the sought points A′, A″. From the figure the roots A′ A : m O A, A″ A : m O A are approximately —.3 and —1.7.

(b.) Roots of $2x^2 + 2x + 4 = 0$. [Fig. II.]

Here $OA = 2$, $AB = 2m$, $BC = 4m^2 = -4$. The circle on O C as diameter does not cut A B and the roots are imaginary. Since O A′ A, A′ C B are similar, therefore A′ is equally distant from A and B, and that distance is mean proportional between O A and C B. A circle with this mean proportional as radius and center at A or B will therefore cut the perpendicular erected at the middle point (M) of A B in the sought points A′, A″. The circle with center at M and cutting the circle on O C as diameter at right angles also passes through these points. Conceiving the step m. O A drawn from A′ we see that M A and A′ M, A″ M are the real and imaginary components of the roots. The roots given by A′ and A″ are by the figure $-\frac{1}{2} - 1.3m$ and $-\frac{1}{2} + 1.3m$.

(c.) Real root of $2x^3 + 4x^2 + 8 \quad 4 = o$.

We have O A = 2, A C = 4 m, B C = $8m^2 = -8$, C D = $4m^3 = -4m$. The circuit O A′ B′ D was drawn by aid of transparent paper turned round a pin with cross section paper underneath, after the manner of Lill's wooden and ground glass discs. The root, A′ A : m O A = tan A′ O A, may be read off from the cross section paper to several decimal places. It is here — .64....

O A′ B′ D is the circuit for the quadratic equation that gives the remaining pair of roots of the cubic. The circle on O D as diameter will not cut A′ B′ so that these roots are imaginary.

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## On some theorems of integrations in quaternions. By A. S. Hathaway.

There are certain identities among volume, surface and line integrals of a quaternion function $q = f(h)$ that include as special cases the well known theorems of Green and Stokes, that are so often employed in mathematical physics. These indentities were first demonstrated by Prof. Tait by the aid of the physical principles usually employed in forming the so-called "Equation of Continuity." [See Tait's Quatermous, third ed., ch. XII J.]

If dh $d_1h, d_2h$ be non-coplanar differentials of the vector h, the theorems may be written:

(1) $-\iiint S\,dh\,d_1h\,d_2h . \nabla q = \iint \mathbf{V}\, dh\,d_1h . q$

(The surface integral extends over the boundary of the volume integral and $\mathbf{V} dh\,d_1h$ is an outward facing element of the surface.)

(2) $\iint \mathbf{V} (\mathbf{V} dh\,d_1h . \nabla) . q = \int dh\,q$

(The line integral extends over the boundary of the surface integral in the positive direction as given by the vector areas $\mathbf{V}\, dh\,d_1h$.)

These theorems are analogous to the elementary theorem,

(3) $\int_A^B dq = q_B - q_A$ or in quaternion notation,

$-\int S\,dh\nabla . q = q$